HOLOMORPHIC DYNAMICS

WE STUDY THE DYNAMICS OF A HOLOMORPHIC MAPPING $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$.

(1) CASE $n=1$ (I.E. $f : \mathbb{C} \to \mathbb{C}$ ENTIRE).

(a) CASE: f IS $(1,1)$.

$$f(z) = \alpha z + \beta$$

DYNAMICS TRIVIAL:

$\alpha = 1,\ \beta = 0 \Longrightarrow$ EVERYTHING FIXED

$\alpha = 1,\ \beta \neq 0 \Longrightarrow$ EVERYTHING $\longrightarrow \infty$

$\alpha \neq 1 \Longrightarrow$ WRITE $z = w + \dfrac{\beta}{1-\alpha}$

$$f(z) = g(w) + \frac{\beta}{1-\alpha}$$

THEN $g(w) = \alpha w$

$|\alpha| < 1 \Longrightarrow$ EVERYTHING $\longrightarrow 0$

$|\alpha| > 1 \Longrightarrow$ EVERYTHING (EXCEPT $w=0$) $\longrightarrow \infty$

$|\alpha| = 1 \Longrightarrow$ RATIONAL & IRRATIONAL ROTATIONS

(b) CASE: f IS (2,1)

$$f(z) = \alpha z^2 + \beta z + \gamma$$

IN NON-TRIVIAL CASES, $z = aw + b$, $f(z) = ag(w) + b$, FOR SUITABLE a, b REDUCES TO:

$$g(w) = w^2 + c$$

JULIA SET

$$J(g) = \{z : \nexists \text{ NBD. } U \text{ OF } z \text{ ON WHICH } \{g^n : n = 1, 2, 3, \ldots\} \text{ IS EQUICTS.}\}$$

MANDELBROT SET

$$M = \{c : J(g) \text{ IS CONNECTED}\}$$

(REF. H.O. PEITGEN/SAUPE)

NOTE g HAS 2 FIXED POINTS, AT MOST 1 ATTRACTING.

$\exists$ ATTRACTING F. PT. $\Leftrightarrow$ c IS INSIDE CARDIOID.

M IS CONNECTED & COMPACT

OPEN QUESTION: IS M LOCALLY CONNECTED?

(c) CASE: f IS $(3,1)$

REDUCE TO: $f(z)=z^3-3a^2z+b$.

ANALOGE OF M IS THE CONNECTEDNESS LOCUS

$$\mathcal{C}_3=\{(a,b): J(f) \text{ IS CONNECTED}\}$$

WHICH IS COMPACT, CONNECTED & NOT LOCALLY CONNECTED.

REF: BRANNER/HUBBARD, ACTA MATH 1988.

(d) SOME $(\infty,1)$ FUNCTIONS ARE INTERESTING.

REF: WORK OF I.N. BAKER & P.J. RIPPON ON THE DYNAMICS OF $f(z)=e^{cz}$.

(2) CASE $F: \mathbb{C}^n \to \mathbb{C}^n$, $n>1$

(a) CASE: F IS (1,1) THIS IS HIGHLY NONTRIVIAL!

IT IS EASY TO GET BIJECTIVE, BIHOLOMORPHIC F (WE CALL THESE AUTOMORPHISMS). FOR EXAMPLE

$$F(z_1,\dots,z_n) = (w_1,\dots,w_n)$$

WHERE

$$\begin{aligned} w_1 &= k_1 z_1 + f_1(z_2,\dots,z_n) \\ w_2 &= k_2 z_2 + f_2(w_1, z_3,\dots,z_n) \\ w_3 &= k_3 z_3 + f_3(w_1, w_2, z_4,\dots,z_n) \\ &\cdots\cdots\cdots \\ w_n &= k_n z_n + f_n(w_1, w_2,\dots,w_{n-1}) \end{aligned}$$

THE FUNCTIONS $f_i : \mathbb{C}^{n-1} \to \mathbb{C}$ BEING HOLOMORPHIC AND THE CONSTANTS k_i NONZERO.

THIS IS A COMPOSITION OF MULTIPLES OF SHEARS:

$$\begin{aligned} w_i &= z_i + f(z_1,\dots,z_{i-1}, z_{i+1},\dots,z_n) \\ w_j &= z_j \quad (j \neq i) \end{aligned}$$

IS A SHEAR IN THE DIRECTION e_i (ROSAY/RUDIN)

USING SHEARS ROSAY & RUDIN PROVE:

(1) $\forall$ DISCRETE $\{p_j\}_{j=1}^{\infty} \subseteq \mathbb{C}^n$, $n>1$ WITHOUT REPETITIONS $\forall \{w_j\}_{j=1}^{\infty} \subseteq \mathbb{C}^n$ $\exists F: \mathbb{C}^n \to \mathbb{C}^n$ HOLOMORPHIC (& $J(F) \equiv 1$ BUT NOT NEC. $\in Aut(\mathbb{C}^n)$) WITH $F(p_j) = w_j$ $\forall j$.
(MITTAG-LEFFLER INTERPOLATION PROBLEM)

(2) $\forall$ COUNTABLE, DENSE $X, Y \subseteq \mathbb{C}^n$ $(n>1)$
$\exists F \in Aut(\mathbb{C}^n)$ S.T. $F(X) = Y$.
BUT (3) $\exists$ DISCRETE $D \subseteq \mathbb{C}^n$ S.T. THE ONLY $F \in Aut(\mathbb{C}^n)$ WITH $F(D) = D$ IS THE IDENTITY (D IS RIGID).

THEY DEFINE $E \subseteq \mathbb{C}^n$ IS TAME IF $\exists F \in Aut(\mathbb{C}^n)$ MAPPING E ONTO AN ARITHMETIC PROGRESSION.
EVERY INFINITE DISCRETE SET CAN BE MAPPED ONTO AN A.P. BY AN INJECTIVE HOLOMORPHIC $F: \mathbb{C}^n \to \mathbb{C}^n$ WITH $J(F) \equiv 1$, BUT NOT NEC. BY AN AM..

AN A.P. CAN BE PERMUTED IN ANY WAY BY A SUITABLE AM., SO THE SAME IS TRUE OF ANY TAME SET; (C.F. (3) ABOVE).
EVERY INFINITE DISCRETE SET IS THE UNION OF 2 TAME SETS.

Ranges of entire holomorphic mappings $F:\mathbb{C}^n \to \mathbb{C}^n$.

$\underline{n=1}$ Picard's Theorem $F(\mathbb{C}) = \mathbb{C}$ or $\mathbb{C}\setminus\{z_0\}$ or $\{z_0\}$.

$\underline{n>1}$ Obvious Conjecture

$F(\mathbb{C}^n) = \mathbb{C}^n$ or $\mathbb{C}^n \setminus$ lower diml. variety

or lower dimensional variety.

E.G. $(z_1, z_2) \mapsto (e^{z_1}, e^{z_2})$ range $\{(w_1, w_2) : w_1 \neq 0, w_2 \neq 0\}$.

$(z_1, z_2) \mapsto ((z_1+z_2), (z_1+z_2)^2)$ range $\{(w_1, w_2) : w_2 = w_1^2\}$.

The "obvious conjecture" is FALSE.

Theorem (Bieberbach, Fatou) $\exists F : \mathbb{C}^2 \to \mathbb{C}^2$ holomorphic with $J(F)$ nowhere zero (so $F(\mathbb{C}^2)$ open) with $F(\mathbb{C}^2)$ not dense in $\mathbb{C}^2$.

Method Suppose $U \subseteq \mathbb{C}^n$ with $F : \mathbb{C}^n \twoheadrightarrow U$ biholomorphic

For $k \in \mathbb{C}$, $|k| > 1$, let $B_k \in Aut(\mathbb{C}^n)$, $B_k(\underset{\sim}{z}) = k\underset{\sim}{z}$.

This induces $\theta \in Aut(U)$ by

$$\theta = F \circ B_k \circ F^{-1}$$

with fixed point $p = F(0)$, $\theta^{-j}(z) \to p \quad \forall z \in U$

and $\theta'(p) = kI = B_k$.

CONVERSELY IF U OPEN $\subseteq \mathbb{C}^n$ AND $\theta \in Aut(U)$ WITH FIXED POINT p AND $\theta^{-j}(z) \to p$ $(z \in U)$ AND $\theta'(z_0) = kI$, $|k|>1$, THEN $\exists F : \mathbb{C}^n \twoheadrightarrow U$ BIHOLOMORPHIC WITH

$$\theta \circ F = F \circ B_k .$$

CONSTRUCTION

$$F(z) = \lim_{j\to\infty} \theta^j(p + k^{-j} z) \qquad (z \in \mathbb{C}^n).$$

THEOREM SUPPOSE $\theta \in Aut(\mathbb{C}^n)$ FIXES $p \in \mathbb{C}^n$ AND THE EIGENVALUES $\lambda_1, \dots, \lambda_n$ OF $\theta'(p)$ SATISFY $|\lambda_j|>1$ $(1 \leq j \leq n)$.
LET
$$W^u(p) = \{z \in \mathbb{C}^n : \lim_{k\to\infty} \theta^{-k}(z) = p\}$$
THE UNSTABLE SET OF p.
THEN $\exists$ POLYNOMIAL AUTOMORPHISM G OF $\mathbb{C}^n$ AND A BIHOLOMORPHIC $F : \mathbb{C}^n \twoheadrightarrow W^u(p)$ SUCH THAT

$$\theta \circ F = F \circ G .$$

GENERALLY, $G = \theta'(p)$, BUT IF THERE ARE RESONANCES

$$\lambda_i = \lambda_1^{m_1} \dots \lambda_n^{m_n}, \quad m_1, \dots, m_n \geq 0, \quad m_1 + \dots + m_n \geq 2,$$

THEN $G = \theta'(p) +$ NONLINEAR TERMS.
THIS G IS A NORMAL FORM FOR θ NEAR p.

EXAMPLE

LET $\theta(x,y) = (a^2x + y^2, ay)$,

WHERE $|a|>1$, SO F.PT. $p=0$, WITH $\lambda_1 = a^2$, $\lambda_2 = a$.

SUPPOSE $\exists f: \mathbb{C}^2 \to \mathbb{C}^2$ WITH $f(0)=0$, $\det f'(0) \neq 0$ AND

$$\theta'(0)^{-1} \circ f \circ \theta = f.$$

THEN $$f(a^2x + y^2, ay) = (a^2, a) f(x,y)$$

IF $f(x,y) = (\phi(x,y), \psi(x,y))$ THEN

$$\phi(a^2x+y^2, ay) = a^2\phi(x,y). \qquad (1)$$

DIFFERENTIATING (1) W.R.T. y,

$$2y\,\phi_x(a^2x+y^2, ay) + a\,\phi_y(a^2x+y^2, ay) = a^2\phi_y(x,y). \qquad (2)$$

PUTTING $x=y=0$ IN (2) $\Longrightarrow \phi_y(0,0) = 0$.

DIFF. (2) W.R.T. y & PUTTING $x=y=0 \Longrightarrow \phi_x(0,0) = 0$.

$$\phi_x(0,0) = 0 = \phi_y(0,0) \Longrightarrow \det f'(0) = 0 \quad ※.$$

PROPERTIES OF THE RANGE OF HOLOMORPHIC F

IF $F: \mathbb{C}^n \to \mathbb{C}^n$ HAS $J(F) \not\equiv 0$, THEN:

(1) $\mathrm{conv}(F(\mathbb{C}^n)) = \mathbb{C}^n$;

(2) $F(\mathbb{C}^n)$ MISSES AT MOST n HYPERPLANES.

IF, FURTHER, F IS INJECTIVE, THEN:

(3) $F(\mathbb{C}^n)$ IS PSEUDOCONVEX;

(4) $\mathrm{vol.}(F(\mathbb{C}^n)) = \infty$; (THIS NEED NOT HOLD IF F IS NOT ASSUMED INJECTIVE — EXAMPLE OF ROSAY & RUDIN).

(5) $\mathbb{C}^n \setminus F(\mathbb{C}^n)$ IS UNBOUNDED UNLESS F IS SURJECTIVE.

ROSAY & RUDIN SHOWED THAT, ALTHOUGH EVERY TAME SET E IS AVOIDABLE BY AN INJECTIVE F (I.E. $F(\mathbb{C}^n) \cap E = \emptyset$), $\exists$ DISCRETE D S.T. $F(\mathbb{C}^n) \cap D \neq \emptyset$ FOR EVERY $F: \mathbb{C}^n \to \mathbb{C}^n$ WITH $J(F) \not\equiv 0$.

MOREOVER, $\exists$ TAME SETS NOT AVOIDABLE BY F WITH $J(F) \equiv 1$.

HENCE $\exists\, \Omega \subseteq \mathbb{C}^n$ $(n>1)$ S.T. $\Omega = F(\mathbb{C}^n)$ FOR AN INJECTIVE F, BUT NOT FOR AN INJECTIVE F WITH $J(F) \equiv 1$.

EXAMPLE: BIEBERBACH'S FUNCTION.

DEFINE THE AUTOMORPHISM θ BY $\theta(x,y)=(u,v)$ WHERE

$$u = kx + j(y),$$
$$v = ky + j(u),$$
$$(\ |k|>1 \text{ AND } j(X) \text{ HAS NO TERMS OF ORDER } <2).$$

LET $k:=4$, $j(X):=2X^5-5X^2$. THEN G IS BIEBERBACH'S FUNCTION.

KODAIRA'S FUNCTION IS DEFINED IN THE SAME WAY, WITH $|k|>1$, $j(X)=(k-1)(\sin X - X)$.
($k=2$, $j(X)=\sin X - X$ IN THE COMPUTER PICTURES).
KODAIRA'S θ HAS FIXED POINTS $(2r\pi, 2r\pi)$ $(r\in\mathbb{Z})$.
THE SETS $U_r:=\{z\in\mathbb{C}^2: \theta^{-n}(z)\to(2r\pi,2r\pi)\}$ $(r\in\mathbb{Z})$ ARE DISJOINT RANGES OF ENTIRE FUNCTIONS DIFFERING BY TRANSLATIONS BY INTEGER MULTIPLES OF $(2\pi,2\pi)$.

FATOU'S FUNCTION COMES FROM

$$\theta:(x,y)\longmapsto(y,\ 2x+4xy-3y^2)$$

$\theta(x,-\frac{1}{2})=(-\frac{1}{2},-\frac{3}{4})$ $(x\in\mathbb{C})$ BUT WE STILL GET A (NON-INJECTIVE) $G:\mathbb{C}^2\twoheadrightarrow\{z\in\mathbb{C}^2: \exists z_n\to 0 \ \ z=\theta^n(z_n)\}$.

HOW SMALL CAN $F(\mathbb{C}^n)$ BE? F BIHOLOMORPHIC.

THE BIEBERBACH EXAMPLE HAS

$$F(\mathbb{C}^2) \subseteq \{(x,y) : |x| < 2 \;\; \& \;\; |x| < |y|\},$$

AND HENCE

$$F(\mathbb{C}^2) \subseteq \{(x,y) : |y - j(x)| < \max\{4|x|, 7\}\}.$$

ANOTHER EXAMPLE (J.E.) HAS

$$F(\mathbb{C}^2) \subseteq \{(x,y) : \operatorname{Re}(x) < 0 \text{ OR } \operatorname{Re}(y) < 0\}$$

NISHIMURA, COUTY, ROSAY & RUDIN

$$\overline{F(\mathbb{C}^2)} \cap \{(x,y) : x = 0\} = \emptyset$$

WHY DO SOME SECTIONS THROUGH $F(\mathbb{C}^n)$ SHOW SMOOTH BOUNDARIES?

STABLE MANIFOLD THEOREM IF $\theta \in Aut(\mathbb{C}^n)$ WITH A FIXED POINT p WHICH IS HYPERBOLIC, THEN

$$W^s(p) = \{z \in \mathbb{C}^n : \theta^j(z) \to p \text{ AS } j \to \infty\}$$

THE STABLE MANIFOLD OF p

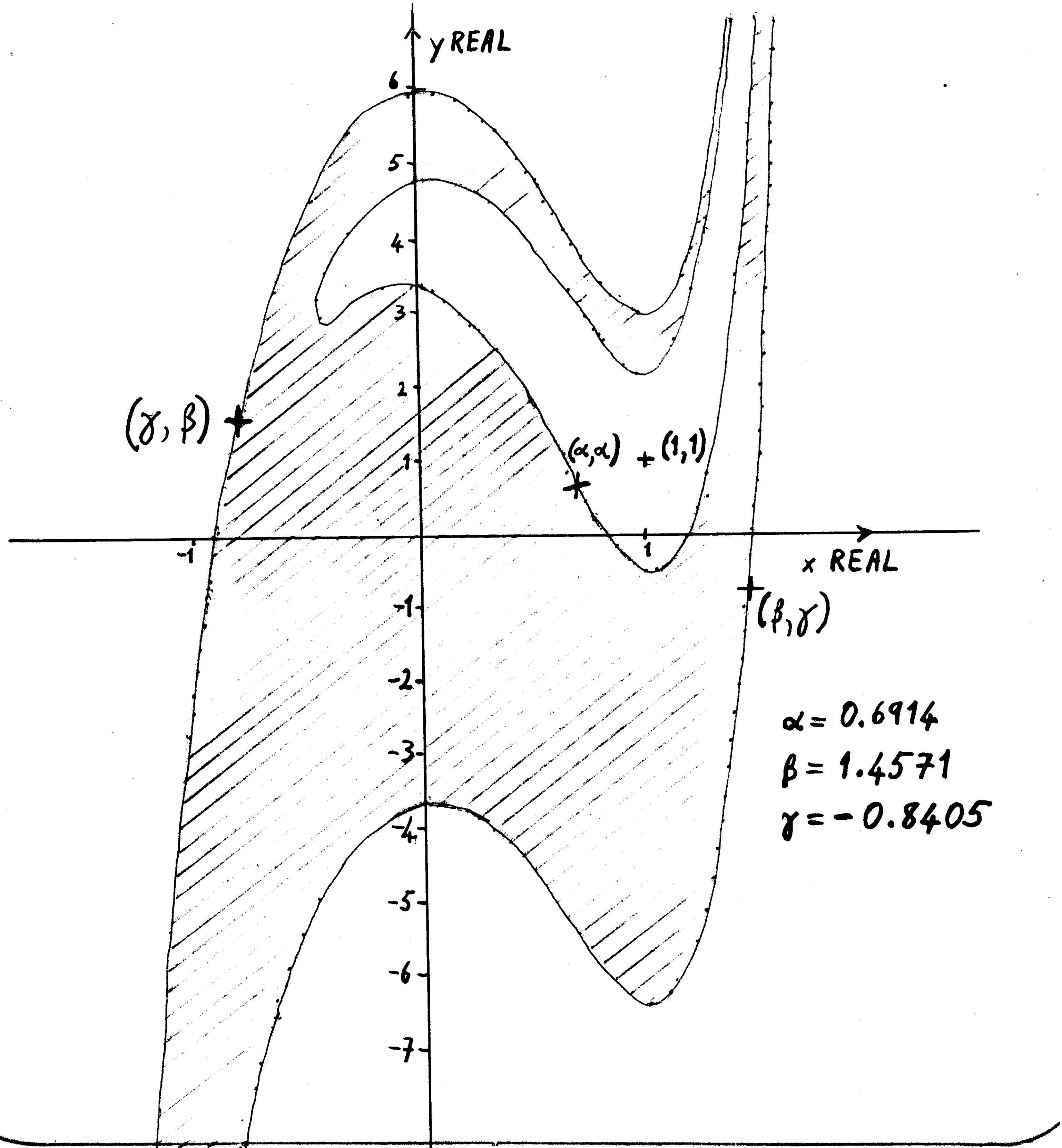
AND

$$W^u(p) = \{z \in \mathbb{C}^n : \theta^{-j}(z) \to p \text{ AS } j \to \infty\}$$

THE UNSTABLE MANIFOLD OF p,

ARE COMPLEX SUBMANIFOLDS OF $\mathbb{C}^n$.

OTHER FEATURES OF THE PICTURES OF $F(\mathbb{C}^2)$'s ARE PROBABLY EXPLICABLE IN TERMS OF FULLER PICTURES OF THE DYNAMICS OF θ. SHOWING ALL BASINS OF ATTRACTION HELPS.



y REAL
x REAL
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-7
-1
1
(γ, β)
(α, α)
$(1,1)$
(β, γ)
$\alpha = 0.6914$
$\beta = 1.4571$
$\gamma = -0.8405$