Varieties of Banach algebras
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[This document and a preprint of my latest paper on the subject may be found at `http://www.sheffield.ac.uk/~pm1pgd/research/p36.html`]

Algebraic Motivation

Consider linear associative algebras over the complex field.

Definition 1 A *variety* of algebras is a (non-empty) class of algebras closed under subalgebras, quotients, products and isomorphic images.

Definition 2 Let $p(X_1, \ldots, X_n)$ be a polynomial (unless otherwise specified, this will mean a polynomial in non-commuting indeterminates, without constant term). We say that an algebra A *obeys the law* $p = 0$ iff

$$p(x_1, \ldots, x_n) = 0 \qquad (x_1, \ldots, x_n \in A).$$

Example 1 An algebra "obeys the law $XY - YX = 0$" iff it is commutative.

Theorem 1 (Birkhoff's Theorem) *For a class $\mathcal{V}$ of algebras the following are equivalent:*

1. *$\mathcal{V}$ is a variety (closed under subalgebras, quotients, products and isomorphic images);*
2. *For some set P of polynomials*

$$\mathcal{V} = \{A : A \textit{ obeys the law } p = 0 \quad \forall p \in P\}.$$

Varieties of Banach algebras

Definition 3 If $\{A_\lambda\}_{\lambda\in\Lambda}$ is a family of Banach algebras, their *product* is the Banach algebra

$$\prod_\lambda A_\lambda = \{x = (x_\lambda)_{\lambda\in\Lambda} : \|x\| = \sup_\lambda \|x_\lambda\| < \infty\}.$$

Definition 4 A (non-empty) class $\mathcal{V}$ of Banach algebras is a *variety* if it is closed under taking: (a) closed subalgebras, (b) quotients by closed ideals, (c) products in the sense just described, and (d) images under isometric isomorphisms.

When we speak of a *polynomial*, we shall mean a polynomial over the complex field, in finitely many non-commuting indeterminates, without constant term. (Polynomials with constant term play an analogous rôle in the theory of varieties of unital Banach algebras.)

If $p(X_1, \ldots, X_n)$ is a polynomial and A is a Banach algebra, we denote by A_1 the closed unit ball of A and define

$$\|p\|_A = \sup\{\|p(x_1, \ldots, x_n)\| : x_1, \ldots, x_n \in A_1\}.$$

Defining

$$\|p\|_1 = \Sigma(\text{moduli of the coefficients of } p),$$

we have $\|p\|_A \leqslant \|p\|_1$ for all polynomials p and Banach algebras A.

Theorem 2 (Birkhoff for Banach algebras) *A class $\mathcal{V}$ of Banach algebras is a variety if and only if there is a non-negative real-valued function $p \mapsto K_p$ on the set of all polynomials such that*

$$\mathcal{V} = \{A : \|p\|_A \leqslant K_p \quad \forall \text{ polynomials } p\}.$$

Essentially, this says that varieties are defined by polynomial inequalities. The two extreme cases are worth noting. First, if $K_p = 0$ then saying $\|p\| \leqslant K_p$ is equivalent to saying $p \equiv 0$ in A. Secondly, if $K_p \geqslant \|p\|_1$, then the inequality $\|p\|_A \leqslant K_p$ is automatically satisfied, and therefore represents no constraint on the algebra A. Thus a class defined by any set of polynomial identities and inequalities can be portrayed as being defined by a function $p \mapsto K_p$.

In [3] we have given a generalization of the original Birkhoff Theorem to universal algebras with relations from which this Banach algebra version may be derived.

Remark 1 For a class of Banach algebras $\mathcal{V}$ and a polynomial p, if

$$\|p\|_{\mathcal{V}} := \sup\{\|p\|_A : A \in \mathcal{V}\}.$$

then Theorem 2 says that $\mathcal{V}$ is a variety iff

$$\mathcal{V} = \{A : \|p\|_A \leqslant \|p\|_{\mathcal{V}}\},$$

Example 1: Q-algebras

Definition 5 A Banach algebra A is a **uniform algebra** it is a closed subalgebra of $C(X)$ for some X.

Definition 6 (Q-algebras: J. Wermer, [13]) A Banach algebra A is a **Q-algebra (IQ-algebra)** if it is isomorphic (isometrically isomorphic) to the quotient of a uniform algebra by a closed ideal.

Example 2 The class IQ of all IQ-algebras is a variety. In fact, it is the smallest variety containing $\mathbb{C}$: the **variety generated by** $\mathbb{C}$. It is therefore defined by the polynomial norm inequalities

$$\|p\|_A \leqslant \|p\|_{\mathbb{C}} = \sup\{|p(z_1, \ldots, z_n)| : |z_i| \leqslant 1\}.$$

This is just Craw's Lemma for IQ-algebras.

Example 2: Operator algebras

Example 3 Let R, IR be the classes of algebras isomorphic (isometrically isomorphic) to closed subalgebras of $\mathcal{B}(H)$ for H a (complex) Hilbert space ("operator algebras").

(WARNING: these subalgebras are not necessarily self-adjoint.)

Theorem 3 (Bernard) *The class IR is closed under quotients.*

It is then easy to see that IR is a variety: the variety generated by $\mathcal{B}(H)$, where H is separable Hilbert space.

Corollary 1 *A Banach algebra A is isometrically isomorphic to an operator algebra iff*

$$\|p\|_A \leqslant \|p\|_{\mathcal{B}(H)}$$

for all polynomials p.

Remark 2 Since

$$\mathbb{C} \cong \{\lambda I \in \mathcal{B}(H) : \lambda \in \mathbb{C}\},$$

$\mathbb{C}$ is an IR-algebra, so we have

$$\|p\|_{IR} \geqslant \|p\|_{\mathbb{C}} = \|p\|_{IQ};$$

$$IQ \subseteq IR.$$

This inclusion is proper, since IR contains non-commutative algebras, but, more subtly, there are commutative IR algebras which are not Q-algebras (Varopoulos).

In order to decide whether a given algebra is isometrically isomorphic to an operator algebra we are asked to look at all polynomials.

Question can we reduce this?

Is it sufficient to consider

- only polynomials in one variable (the classical von Neumann inequality)? Answer: **NO** (PGD 1995) but that answer depended strongly on the fact that we are considering non-unital polynomials.
- only polynomials in less than a certain finite number of variables?
- only polynomials of less than a certain degree?
- only finitely many polynomials?

Proof of Theorem 2 The 'if' part is straightforward. We prove the 'only if'.

For every polynomial p and every $\varepsilon > 0$, we find $C(p, \varepsilon) \in \mathcal{C}$ such that

$$\|p\|_{C(p,\varepsilon)} > \|p\|_{\mathcal{C}} - \varepsilon.$$

If M is the product of all the algebras $C(p, \varepsilon)$, then $M \in \mathcal{C}$ and

$$\|p\|_M = \|p\|_{\mathcal{C}}$$

for every polynomial p.

It is clear that

$$\mathcal{C} \subseteq \{A : \|p\|_A \leqslant \|p\|_{\mathcal{C}} \text{ for all } p\}.$$

We prove the reverse inclusion. Suppose A is a Banach algebra such that, for all polynomials p,

$$\|p\|_A \leqslant \|p\|_{\mathcal{C}} = \|p\|_M.$$

Put

$$X = \Big(M_1\Big)^{A_1},$$

the set of all functions from A_1 into M_1.

Let $\Gamma = M^X$, meaning the algebra of all bounded functions from X into M, this is the Banach algebra product of X copies of M, so $\Gamma \in \mathcal{C}$. For $a \in A_1$, define $\gamma_a \in \Gamma$ by

$$\gamma_a(x) = x(a) \qquad (x \in X).$$

Then $\|\gamma_a\| = 1$. Let

$$U_0 = \text{subalgebra of } \Gamma \text{ generated by } \{\gamma_a : a \in A_1\},$$

and define a homomorphism

$$\theta : U_0 \to A \qquad \theta(\gamma_a) = a \qquad (a \in A_1).$$

A typical element of U_0 is

$$u = p(\gamma_{a_1}, \ldots, \gamma_{a_n})$$

for some polynomial p and distinct $a_1, \ldots, a_n \in A_1$. Then

$$\theta(u) = p(a_1, \ldots, a_n).$$

We show that θ is well-defined ($u = 0 \Rightarrow \theta(u) = 0$) and continuous with $\|\theta\| \leqslant 1$:

$$\begin{aligned}
\|\theta(u)\| &= \|p(a_1, \ldots, a_n)\| \\
&\leqslant \|p\|_A \\
&\leqslant \|p\|_M \\
&= \sup\{\|p(x(a_1), \ldots, x(a_n))\| : x \in X\} \\
&= \sup\{\|u(x)\| : x \in X\} \\
&= \|u\|.
\end{aligned}$$

Let $U = \overline{U_0}$. Then $\theta : U \to A$ is surjective, since $\theta : U_0 \to A$ is already surjective. To show that

$$A \cong U/\ker\theta, \text{ isometrically,}$$

we observe that for $a \in A$ with $\|a\| = 1$ we have $a = \theta(\gamma_a)$ and $\|\gamma_a\| = 1$. This, with $\|\theta\| \leqslant 1$ proves the natural isomorphism to be isometric.

We have that the closure of $\mathcal{C}$ under

$$\begin{array}{rcl} \text{products} & \Rightarrow & \Gamma \in \mathcal{C} \\ \text{closed subalgebras} & \Rightarrow & U \in \mathcal{C} \\ \text{quotients} & \Rightarrow & U/\ker\theta \in \mathcal{C} \\ \text{isometric isomorphisms} & \Rightarrow & A \in \mathcal{C}, \end{array}$$

as required.

Corollary 2 *Every variety of Banach algebras is singly generated.*

The algebra M in the above proof is a generator of the variety $\mathcal{C}$.

Semivarieties

To accommodate Q and R, the classes of Banach algebras isomorphic (i.e. bicontinuously isomorphic) to some IQ-algebra, IR-algebra, respectively, we introduce the notion of a "semivariety".

Definition 7 A **semivariety** is a class $\mathcal{S}$ of Banach algebras such that there is a variety $\mathcal{V}$ of Banach algebras such that $\mathcal{S}$ is the class of all Banach algebras (bicontinuously) isomorphic to some algebra in $\mathcal{V}$.

Theorem 4 (PGD, [2]) *For a non-empty class $\mathcal{S}$ of Banach algebras, the following are equivalent:*

(i) $\mathcal{S}$ is a semivariety of Banach algebras;

(ii) there is a function $p \mapsto K(p)$ from the set of all polynomials to the non-negative reals such that $\mathcal{S}$ is precisely the class of Banach algebras A for which there exist $M, \delta > 0$ such that, for every polynomial $p(X_1, \dots, X_n)$ and all $x_1, \dots, x_n \in A_\delta$ $(= \{x \in A : \|x\| \leqslant \delta\})$,

$$\|p(x_1, \dots, x_n)\| \leqslant M\, K(p);$$

(iii) ditto, but for homogeneous polynomials only.

Examples of Semivarieties

1. The semivariety of all Q-algebras is the class of Banach algebras A for which there exist $M, \delta > 0$ such that, for every polynomial p,

$$\|p(x_1, \dots, x_n)\| \leqslant M\|p\|_{\mathbb{C}} \qquad (x_1, \dots, x_n \in A_\delta).$$

2. The semivariety of all R-algebras is the class of Banach algebras A for which there exist $M, \delta > 0$ such that, for every polynomial p,

$$\|p(x_1, \ldots, x_n)\| \leqslant M\|p\|_{\mathcal{B}(H)} \quad (x_1, \ldots, x_n \in A_\delta).$$

The Q-algebras form a proper subset of the commutative R-algebras (Varopoulos [12] — a very deep result).

Lots of varieties

M. H. Faroughi has shown that the lattice of varieties of Banach algebras contains uncountable chains and antichains...

as does the lattice of semivarieties (much harder).

Algebraically-defined varieties

Most varieties are not semivarieties — i.e. they are not closed under isomorphisms. However, there is an obvious way to construct some which are.

Definition 8 If $\mathcal{V}$ is a variety defined by polynomial identities, that is, if

$$\mathcal{V} = \{A : A \text{ is a Banach alg. } \& \ \|p\|_A = 0 \ (p \in \mathcal{P})\}$$

for some set $\mathcal{P}$ of polynomials, then $\mathcal{V}$ is said to be *algebraically defined*, (abbreviated AD).

It is clear that an AD-variety is closed under isomorphisms and is therefore a semivariety. Thus AD $\Rightarrow$ closed under isomorphism.

In fact, the converse is true: every variety which is closed under isomorphisms is AD, (PGD, to appear in Math. Proc Camb. Philos. Soc.), but the proof required the development of a theory of graded Banach algebras.

B*-algebras

A similar theorem holds for Banach *-algebras, but the polynomials are polynomials in the variables and their adjoints and the subalgebras are all *-subalgebras.

B*-algebras form a variety: what are the defining laws?

Theorem 5 (Corollary of Vidav–Palmer) *A Banach *-algebra A (with 1) is a B*-algebra iff*

$$\left.\begin{array}{rcl} \|\exp(it(x+x^*))\| & \leqslant & 1 \\ \|\exp(t(x-x^*))\| & \leqslant & 1 \end{array}\right\} \qquad (x \in A,\ t \in \mathbb{R}). \tag{1}$$

Let

$$E_n(X) = \sum_{j=0}^{n} \frac{X^j}{j!} \qquad (n = 1, 2, 3, \ldots)$$

and

$$\varepsilon_{n,t} = \sum_{j=n+1}^{\infty} \frac{|2t|^j}{j!} \qquad (n = 1, 2, 3, \ldots).$$

Then (1) is equivalent to

$$\left.\begin{array}{rcl} \|E_n(it(x+x^*))\| & \leqslant & 1+\varepsilon_{n,t} \\ \|E_n(t(x-x^*))\| & \leqslant & 1+\varepsilon_{n,t} \end{array}\right\} \qquad (x \in A_1) \tag{2}$$

for all $t \in \mathbb{R},\ n \in \mathbb{N}$.

Problem area 1 — Unital Banach algebras

Definition 9 A **unital Banach algebra** is a Banach algebra with an identity element e such that $\|e\| = 1$.

We can easily rewrite the definition of variety for unital Banach algebras. A **variety of unital Banach algebras** is a class of unital Banach algebras closed under products, quotients by closed ideals and unital subalgebras (meaning: subalgebras containing the identity element of the original algebra).

This definition fits with a more general theory [3], and we get a version of Birkhoff's Theorem (using unital polynomials). Unfortunately, it is not clear how a theory of semivarieties should go. The problem is that not all isomorphisms are isometries, but we do not have a plentiful supply of non-isometric isomorphisms as we do in general Banach algebra theory. The characterization of semivarieties relies heavily on the fact that one can take a Banach algebra, multiply the norm by any constant $\lambda \geqslant 1$, and obtain an isomorphic Banach algebra. The requirement $\|e\| = 1$ kills this technique for unital algebras.

Question 1 Consider the Q-algebra $C^1[0,1]$ as a unital Banach algebra. Is there an isomorphism between it and an algebra in the unital variety generated by $\mathbb{C}$?

Since $C^1[0,1]$ is a Q-algebra, there is an isomorphism between it and an IQ-algebra, and this IQ algebra has an identity, but the norm of that identity may be greater than 1.

Any IQ-algebra with an identity of norm 1 is necessarily in the unital variety generated by $\mathbb{C}$. Therefore, what we are asking is just whether $C^1[0,1]$ is isomorphic with an IQ-algebra with an identity **of norm 1**.

Given an IQ-algebra with an identity of norm 1, one can find an equivalent norm in which the identity is of norm 1, (use the multiplier norm); however, this will generally destroy the IQ property!

Problem area 2 — The finite basis problem

Definition 10 We say that a variety $\mathcal{V}$ is **finitely based** if it is defined by finitely many laws. That is, there are polynomials $p_1, \ldots, p_n$ and constants $K_1, \ldots, K_n$ such that

$$\mathcal{V} = \{A : \|p_i\|_A \leqslant K_i \qquad (1 \leqslant i \leqslant n)\}.$$

BEWARE: not the same as **finitely generated**: every variety is finitely generated.

Question 2 Are the varieties IQ, IR finitely based?

These are basic questions. To ask whether IQ is finitely based is to ask whether there is a finite set of polynomials inequalities holding in $\mathbb{C}$ from which all other polynomials inequalities holding in $\mathbb{C}$ may be deduced.

Theorem 6 (PGD) *There exists a variety which is not finitely based. (Constructed using laws of the form*

$$\|X_1 \cdots X_n\|_A \leqslant f(n),$$

for a rapidly decreasing function f.)

Remark 3 One can also ask for small, not necessarily finite, sets of laws defining a variety — e.g. polynomials in $< n$ variables or polynomials of degree $< k$.

The problem in algebra

The finite basis problem for groups was posed by B. H. Neumann in his doctoral thesis in 1935 and solved negatively by Vaughan-Lee in 1969. A simple example of a variety without a finite basis is that with basis

$$x_1^8 \equiv 1, \quad (x_1^2 x_2^2)^4 \equiv 1, \ldots, (x_1^2 \ldots x_n^2)^4 \equiv 1, \ldots.$$

This followed various positive partial results such as:

Theorem 7

(R. C. Lyndon) Every variety of nilpotent groups has a finite basis.

The finite basis problem for associative algebras over fields of characteristic zero (Specht's problem) was solved affirmatively by A. R. Kemer [9].

Summary

- The theory of varieties in pure algebra has a sensible analogue in Banach algebra theory.
- There are a number of variations: semivarieties, unital varieties, varieties of B*-algebras
- Two varieties (IQ and IR) are particularly interesting for functional analysis.
- Algebraic problems such as the finite basis problem have analogues which are deep questions for functional analysts.

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