Banach algebras satisfying the non-unital von Neumann inequality

P. G. Dixon

11 June 2005

Abstract

There is a Banach algebra satisfying the von Neumann inequality for polynomials in a single variable, without constant term, which is not isomorphic to a norm-closed algebra of operators on a Hilbert space.

1 Introduction.

For a polynomial $p(X)$ in one variable, we write

$$\|p\|_\infty = \sup\{|p(z)| : z \in \mathbb{C},\ |z| \leqslant 1\}.$$

We say that a (complex) Banach algebra A *satisfies the von Neumann inequality* if, for all polynomials $p(X)$,

$$\|p(x)\| \leqslant \|p\|_\infty \qquad (x \in A,\ \|x\| \leqslant 1).$$

Von Neumann showed ([10], see [12]) that the algebra $\mathcal{L}(H)$ of all bounded operators on a Hilbert space H satisfies the von Neumann inequality. It follows that every Banach algebra which is an *operator algebra*, (i.e. which is isometrically isomorphic to a closed, not necessarily self-adjoint, subalgebra of $\mathcal{L}(H)$ for some Hilbert space H), also satisfies the von Neumann inequality. This paper addresses the converse question: is it true that every Banach algebra which satisfies the von Neumann inequality is an operator algebra?

At this point, we must split the question into two: either we allow polynomials with constant term, in which case we can discuss only unital Banach algebras, or we restrict to polynomials without constant term, when we can discuss all Banach algebras. In this paper, we take the latter course and show that there is a commutative Banach algebra which satisfies the non-unital von Neumann inequality, but is not an operator algebra and is not even bicontinuously isomorphic to an operator algebra.

The question of whether the unital von Neumann inequality characterizes (unital) operator algebras remains open. What is known ([6]) is that, even in the non-unital case, a Banach algebra A is an operator algebra if and only if, for every n and every polynomial $p(X_1, \ldots, X_n)$ without constant term,

$$\|p\|_A \leqslant \|p\|_{\mathcal{L}(H)},$$

where $\|p\|_B$ denotes the supremum of $\|p(x_1, \ldots, x_n)\|$ over all $x_1, \ldots, x_n$ in the unit ball of the Banach algebra B. The author's theory of "varieties of Banach algebras" ([7], [8]) gives

1991 *Mathematics Subject Classification* 47D25.

an algebraic perspective on this result. It would be interesting to know whether operator algebras can be characterized by this condition on some finite set of polynomials (the "finite basis problem" for the variety of operator algebras). However, it is not even known whether it suffices to consider polynomials of small degree or polynomials in a small number of variables. The present paper contributes to this project by showing that it is not sufficient to consider just polynomials in a single variable.

2 The main theorem.

For $0 < \beta \leqslant 1$, let A_β be the Banach algebra $\ell^1(\mathbb{N})$ of all summable complex sequences $x = (x_1, x_2, \ldots)$ with convolution multiplication

$$(xy)_i = \sum_{j=1}^{i-1} x_j y_{i-j} \qquad (i = 1, 2, 3 \ldots),$$

but with norm $\|x\| = \beta^{-1}\|x\|_1$. We note that

$$\|x^n\| = \beta^{-1}\|x^n\|_1 \leqslant \beta^{-1}\|x\|_1^n = \beta^{n-1}\|x\|^n \qquad (x \in A,\ n \geqslant 2). \tag{1}$$

Theorem 2.1 *For $0 < \beta \leqslant 1$, the algebra A_β is not (bicontinuously) isomorphic to an operator algebra; but A_β satisfies the non-unital, single variable, von Neumann inequality if, and only if, $0 < \beta \leqslant \frac{1}{3}$.*

Proof. To show that A_β is not bicontinuously isomorphic to an operator algebra, we use an argument of N. J. Young ([14] p.62). Every C*-algebra is Arens regular, every subalgebra of an Arens regular algebra is Arens regular, and every algebra isomorphic to an Arens regular algebra is Arens regular; therefore, every algebra isomorphic to an operator algebra is Arens regular. It is well-known ([4] section 6, also [15] Theorem 2(d)) that $\ell^1(\mathbb{N})$ is not Arens regular. Therefore, A_β is not isomorphic to an operator algebra.

It is interesting to observe that an alternative proof of the weaker statement that A_β is not (isometrically) an operator algebra can be obtained by using Ando's Theorem [1]. This theorem says that if S, T are commuting contractions on a Hilbert space and $p(X, Y)$ is a polynomial in two commuting variables, then

$$\|p(S,T)\| \leqslant \|p\|_\infty \stackrel{\text{def}}{=} \sup\{|p(x,y)| : x, y \in \mathbb{C},\ |x| \leqslant 1,\ |y| \leqslant 1\}.$$

In our language: if A is a commutative operator algebra, then $\|p\|_A \leqslant \|p\|_\infty$ for all polynomials p in two commuting variables.

Consider the polynomial

$$p(X,Y) = X + Y - \frac{1}{4}(X-Y)^2.$$

This has $\|p\|_\infty = 2$; for if $x, y \in \mathbb{C}$ with $|x|, |y| \leqslant 1$, then

$$|x+y|^2 = 2|x|^2 + 2|y|^2 - |x-y|^2 \leqslant 4 - |x-y|^2 \leqslant \left(2 - \frac{1}{4}|x-y|^2\right)^2,$$

so

$$|p(x,y)| \leqslant |x+y| + \frac{1}{4}|x-y|^2 \leqslant 2.$$

However, if, in the algebra $A = A_\beta$, we let

$$\begin{aligned} a &= (0, \beta, 0, 0, 0, \ldots), \\ b &= (0, 0, \beta, 0, 0, \ldots), \end{aligned}$$

then $\|a\| = \|b\| = 1$, but

$$p(a, b) = \left(0, \beta, \beta, -\frac{1}{4}\beta^2, \frac{1}{2}\beta^2, -\frac{1}{4}\beta^2, 0, 0, 0, \ldots\right)$$

and $\|p(a, b)\| = 2 + \beta$. Thus $\|p\|_A > \|p\|_\infty$.

We now prove that for $0 < \beta \leqslant \frac{1}{3}$, the algebra A_β satisfies the non-unital, single variable, von Neumann inequality. Our proof will depend on an old theorem of H. Bohr.

Theorem 2.2 (Bohr, 1914) *Let f be continuous on the closed unit disc and holomorphic in the interior, with Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1).$$

Then

$$\sum_{n=0}^{\infty} |a_n| r^n \leqslant \|f\|_\infty \qquad (0 \leqslant r \leqslant \frac{1}{3}).$$

This was discovered by Bohr [2] with a bound of $\frac{1}{6}$ in place of $\frac{1}{3}$. The bound $\frac{1}{3}$, which is best possible, was obtained independently by M. Riesz, Schur and Wiener, and Wiener's proof is reproduced in [2]. Another proof was discovered by Sidon [11] and Tomić [13].

Let

$$p(X) = a_1 X + a_2 X^2 + \ldots + a_m X^m$$

be a non-unital polynomial in one variable. Applying Bohr's Theorem to the function $f(z) = p(z)/z$, we see that

$$|a_1| + |a_2|\beta + |a_3|\beta^2 + \ldots + |a_m|\beta^{m-1} \leqslant \|f\|_\infty = \|p\|_\infty \qquad (0 \leqslant \beta \leqslant \frac{1}{3}).$$

If $0 < \beta \leqslant \frac{1}{3}$ and $x \in A_\beta$ with $\|x\| \leqslant 1$ then, using (1), we have

$$\|p(x)\| \leqslant \sum_{i=1}^{m} |a_i| \, \|x^i\| \leqslant \sum_{i=1}^{m} |a_i| \, \beta^{i-1} \leqslant \|p\|_\infty.$$

Thus A_β satisfies the non-unital, single variable, von Neumann inequality

The fact that $\frac{1}{3}$ is the best possible constant in Bohr's Theorem is easily seen by looking at the functions

$$\phi_\alpha(z) = \frac{z + \alpha}{1 + \alpha z} \qquad (|z| \leqslant 1).$$

for a parameter $\alpha \in [0, 1)$. Clearly $\|\phi_\alpha\|_\infty = 1$; in fact, $|\phi_\alpha(z)| = 1$ whenever $|z| = 1$. We have

$$\phi_\alpha(z) = \alpha + (1 - \alpha^2)z - \alpha(1 - \alpha^2)z^2 + \alpha^2(1 - \alpha^2)z^3 - \ldots$$

and

$$\alpha + (1 - \alpha^2)r + \alpha(1 - \alpha^2)r^2 + \alpha^2(1 - \alpha^2)r^3 + \ldots = \alpha + (1 - \alpha^2)\frac{r}{1 - \alpha r}$$

which is greater than 1 if $r > 1/(1+2\alpha)$. Letting $\alpha \to 1$ shows that Bohr's result fails for every $r > \frac{1}{3}$.

This argument shows us immediately that the non-unital von Neumann inequality fails in A_β if $\beta > \frac{1}{3}$. Otherwise, the inequality would extend to functions holomorphic on the closed unit disc and vanishing at 0, such as the functions $\psi_\alpha(z) = z\phi_\alpha(z)$ $(0 \leqslant \alpha < 1)$, and we should have

$$\|\psi_\alpha(x)\| \leqslant \|\psi_\alpha\|_\infty = 1 \qquad (x \in A_\beta,\ \|x\| \leqslant 1).$$

However, taking $x = (\beta, 0, 0, 0, \ldots) \in A_\beta$ gives $\|x\| = 1$ but

$$\begin{aligned} \|\psi_\alpha(x)\| &= \alpha + (1-\alpha^2)\beta + \alpha(1-\alpha^2)\beta^2 + \alpha^2(1-\alpha^2)\beta^3 + \ldots \\ &> 1, \end{aligned}$$

for α sufficiently close to 1. This completes the proof of Theorem 2.1. $\Diamond$

An equivalent formulation of this construction, *up to an isometric isomorphism,* is to describe A_β as the weighted sequence algebra $\ell^1(\mathbb{N}, \omega)$ where the weight function ω is given by $\omega(i) = \beta^{i-1}$. The motivation for defining A_β is then clear: the weight is needed to make powers small so that the von Neumann inequality is satisfied. On the other hand, ([5] Theorem (5.3)), too rapidly decreasing a weight would produce a Q-algebra, which is necessarily isomorphic to a operator algebra. The weight chosen is, in fact, rather slowly decreasing.

3 Appendix—proofs of Bohr's Theorem.

As Bohr's theorem is unjustly neglected in the literature on complex analysis, we include in this appendix, which is not intended for publication, a survey of its proofs.

3.1 Wiener's proof

It is well known (from the Cauchy Inequality) that $|a_n| \leqslant \|f\|_\infty$ for all n. By multiplying f by a suitable constant, we may suppose, without loss of generality, that $\|f\|_\infty < 1$ and that $a_0 \geqslant 0$. Let

$$\Phi(z) = \frac{z - a_0}{1 - a_0 z} \qquad (|z| \leqslant 1).$$

Since $0 \leqslant a_0 < 1$, the function Φ is well-defined and in the disc algebra $A(\overline{D})$, with $\|\Phi\|_\infty = 1$. The function $g = \Phi \circ f$ is therefore in $A(\overline{D})$ with $\|g\|_\infty \leqslant 1$. Therefore

$$\left|\frac{a_1}{(1-a_0^2)}\right| = |g'(0)| \leqslant \|g\|_\infty \leqslant 1;$$

i.e. $|a_1| \leqslant 1 - a_0^2$.

Now let n be any integer greater than 1 and let $\omega = e^{2\pi i/n}$. Then the function F defined by

$$F(z) = \frac{1}{n}\left(f(z) + f(\omega z) + \ldots + f(\omega^{n-1}z)\right) = \sum_{m=0}^{\infty} a_{mn} z^{mn}$$

is in $A(\overline{D})$ with $\|F\|_\infty < 1$. Applying the argument above to $F(z^{1/n})$ in place of $f(z)$ shows that $|a_n| \leqslant 1 - a_0^2$.

Now if $0 \leqslant r \leqslant \frac{1}{3}$, then

$$\sum_{n=0}^{\infty} |a_n| r^n \leqslant a_0 + \left(\sum_{n=1}^{\infty} r^n \right) (1 - a_0^2) \leqslant a_0 + \frac{1}{2}(1 - a_0^2) = 1 - \frac{(1 - a_0)^2}{2} < 1.$$

The result follows immediately.

3.2 A proof from Löwner's Theorem

An alternative formulation of Wiener's proof makes use of ideas surrounding the concept of "subordination". We begin by quoting a theorem of K. Löwner.

Theorem 3.1 (Löwner[9], see [3] Theorem 6.34) *Let $\phi(z) = \sum_{n=1}^{\infty} a_n z^n$ and $\psi(z) = \sum_{n=1}^{\infty} b_n z^n$ be holomorphic in the open unit disc D. Suppose that ϕ is one-to-one, $\phi(D)$ is convex and $\psi(D) \subseteq \phi(D)$ (i.e. that "ψ is subordinate to ϕ"). Then*

$$|b_k| \leqslant |a_1| \qquad (k \geqslant 1).$$

We take $\psi = f$ and, as before, we may suppose that $\|f\|_\infty < 1$ and $0 \leqslant a_0 < 1$. Let

$$\phi(z) = \frac{z + a_0}{1 + a_0 z} \qquad (|z| \leqslant 1).$$

This makes ϕ the inverse function to the function Φ defined above. Thus ϕ is one-to-one, $\phi(D) = D$ which is convex and $\psi(D) = f(D) \subseteq \phi(D)$ since $\|f\|_\infty < 1$. Now

$$\phi(z) = a_0 + (1 - a_0^2)z - a_0(1 - a_0^2)z^2 + a_0^2(1 - a_0^2)z^3 - \ldots,$$

so Löwner's Theorem implies that

$$|a_n| \leqslant 1 - a_0^2 \qquad (n \geqslant 1),$$

from which the result follows as before.

3.3 A proof from the Hadamard–Borel–Carathéodory Inequalities

The following proof, obtaining the constant $\frac{1}{6}$ in place of $\frac{1}{3}$, is how the present author first found this theorem. It is similar to Bohr's original proof.

As before, we may suppose that $\|f\|_\infty < 1$ and $0 \leqslant a_0 < 1$ and we have $|a_n| \leqslant 1$ for all n. Let

$$g(z) = f(z) - a_0 = a_1 z + a_2 z^2 + \ldots.$$

Then

$$g(z) \in \{w : |w + a_0| \leqslant 1\} \qquad (|z| \leqslant 1),$$

so, in particular,

$$\sup\{\operatorname{Re} g(z) : |z| \leqslant 1\} \leqslant 1 - a_0.$$

Let

$$M_R(g) = \sup\{|g(z)| : |z| = R\}.$$

Then, by the Hadamard–Borel–Carathéodory Inequalities ([3], Theorem 6.31),

$$M_R(g) \leqslant \frac{2R}{1 - R} \sup\{\operatorname{Re} g(z) : |z| \leqslant 1\} \leqslant \frac{2R(1 - a_0)}{1 - R} \leqslant (1 - a_0)$$

when $0 < R \leqslant \frac{1}{3}$. By the Cauchy Inequality,

$$|a_n|R^n \leqslant M_R(g) \leqslant (1 - a_0) \qquad (n \geqslant 1).$$

Therefore, if $0 \leqslant r \leqslant \frac{1}{6}$, we can write $r = R/2$ and obtain

$$\sum_{n=0}^{\infty} |a_n| r^n \leqslant a_0 + (1 - a_0) \left(\frac{1}{2} + \frac{1}{4} + \dots \right) = 1.$$

This completes the proof.

3.4 The Sidon–Tomić approach.

The following proof is based on those of Sidon and Tomić. Let ϕ_n denoted the argument of a_n. Then, for $0 < \rho < 1$,

$$f(\rho e^{i\theta}) = \sum_{n=0}^{\infty} a_n \rho^n e^{in\theta} = \sum_{n=0}^{\infty} |a_n| \rho^n \exp(i(n\theta + \phi_n)) \qquad (0 \leqslant \theta \leqslant 2\pi).$$

Therefore

$$\operatorname{Re} f(\rho e^{i\theta}) = \sum_{n=0}^{\infty} |a_n| \rho^n \cos(n\theta + \phi_n) \qquad (0 \leqslant \theta \leqslant 2\pi).$$

Since the functions $\cos(n\theta + \phi_n)$ $(n = 0, 1, 2, \dots)$ are orthogonal on $[0, 2\pi]$, we get

$$\begin{aligned} |a_n| &= \lim_{\rho \nearrow 1} \frac{1}{\pi} \int_0^{2\pi} \operatorname{Re} f(\rho e^{i\theta}) \cos(n\theta + \phi_n) \, \mathrm{d}\theta \qquad (n \geqslant 1), \\ |a_0| &= \lim_{\rho \nearrow 1} \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2} \operatorname{Re} f(\rho e^{i\theta}) \, \mathrm{d}\theta. \end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} |a_n| r^n = \lim_{\rho \nearrow 1} \frac{1}{\pi} \int_0^{2\pi} \operatorname{Re} f(\rho e^{i\theta}) \left(\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos(n\theta + \phi_n) \right) \mathrm{d}\theta.$$

Now if $0 < r \leqslant \frac{1}{3}$ then the 'kernel' in this integral is positive: i.e.

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos(n\theta + \phi_n) \geqslant \frac{1}{2} - \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n \geqslant 0.$$

It follows that

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n| r^n &\leqslant \frac{1}{\pi} \int_0^{2\pi} \max_{|z|<1} (\operatorname{Re} f(z)) \left(\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos(n\theta + \phi_n) \right) \mathrm{d}\theta \\ &= \max_{|z|<1} (\operatorname{Re} f(z)) \\ &\leqslant \|f\|_\infty. \end{aligned}$$

References

[1] T. Ando, "On a pair of commutative contractions", *Acta Sci. Math. (Szeged),* **24** (1963), 88–90. MR 27 #5132

[2] H. Bohr, "A theorem concerning power series", *Proc. London Math. Soc.* (2), **13** (1914), 1–5.

[3] R. B. Burckel, "An introduction to classical complex analysis, vol.1", (Birkhäuser, Basel, Mathematische Reihe **64**, 1979).

[4] M. M. Day, "Amenable semigroups", *Illinois J. Math.,* **1** (1957), 509–544.

[5] P. G. Dixon, "Radical Q-algebras", *Glasgow Math. J.,* **17** (1976), 119–126.

[6] P. G. Dixon, "A characterization of closed subalgebras of $B(H)$", *Proc. Edinburgh Math. Soc.,* **20** (1976–77), 215–217.

[7] P. G. Dixon, "Varieties of Banach algebras", *Quart. J. Math. Oxford* (2), **27** (1976), 481–487.

[8] P. G. Dixon, "Classes of algebraic systems defined by universal Horn sentences", *Algebra Universalis,* **7** (1977), 315–339.

[9] K. Löwner, "Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises $|z| < 1$, die durch Funktionen mit nicht verschwindender Ableitung geliefert werden", *Ber. Verhandl. Kön. Sächs. Gesell. Wiss. Leipzig*, **69** (1917), 89–106. FM 46, p.556

[10] J. von Neumann, "Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes", *Math. Nachr.,* **4** (1951), 258–281. MR 13, 254

[11] S. Sidon, "Uber einen Satz von Herrn Bohr", *Math. Z.,***26** (1927), 731–732. (FM **53**,281).

[12] Béla Sz.-Nagy and C. Foiaş, "Harmonic analysis of operators on Hilbert space" (North-Holland, 1970).

[13] M. Tomić, "Sur un théorème de H. Bohr", *Math. Scand.,* **11** (1962), 103–106. (MR **31** #316; Zbl. 109 p.302.)

[14] N. J. Young, "The irregularity of multiplication in group algebras", *Quart. J. Math.* (2), **24** (1973), 59–62.

[15] N. J. Young, "Semigroup algebras having regular multiplication", *Studia Math.,* **47** (1973), 191–196.

Dept. of Pure Mathematics,
University of Sheffield,
SHEFFIELD, S3 7RH,
England.

e-mail: P.Dixon@sheffield.ac.uk