Graded Banach algebras associated with varieties of Banach algebras

by P. G. Dixon (University of Sheffield)

Author's address for correspondence:

Dr. P. G. Dixon,
Department of Pure Mathematics,
University of Sheffield,
SHEFFIELD, S3 7RH,
England.

e-mail: P.Dixon@sheffield.ac.uk

Graded Banach algebras associated with varieties of Banach algebras

P. G. Dixon

29 May 2002

Abstract

The concept of 'varieties of Banach algebras' was introduced by the author (*Quart. J. Math. Oxford* (2), **27** (1976), 481–487). Here we develop the theory, producing analogues of the 'relatively free' objects in varieties of universal algebras. We take as a test question the problem of describing all varieties which are closed under bicontinuous isomorphisms (i.e. varieties which are also semivarieties), and show that this may be answered easily with the aid of these new concepts. (The answer is, as expected, just those varieties which are 'algebraically defined'.)

1 Introduction

Consider the class IR of all Banach algebras which are isometrically isomorphic to norm-closed (not necessarily self-adjoint) algebras of bounded operators on Hilbert space. This class is closed under the operations of taking: (a) closed subalgebras, (b) products (in the ℓ^∞ norm), (c) quotients ([1], see also [5]) and (d) images under isometric isomorphisms. We call a class of Banach algebras closed under (a) – (d) a *variety*; it is the obvious analytic analogue of a variety of rings or a variety of groups, as studied in pure algebra.

In algebra, the most important theorem about varieties is Birkhoff's Theorem [2] of 1935. In the case of varieties of complex associative algebras, this says that a class of algebras $\mathcal{V}$ is a variety if and only if it is defined by polynomial identities, in the sense that there is a set of polynomials $\mathcal{P}$ such that

$$\mathcal{V} = \{A : p(x_1, \ldots, x_n) = 0 \quad (x_1, \ldots, x_n \in A) \text{ for all } p \in \mathcal{P}\}.$$

In [3], we showed that varieties of Banach algebras can be characterized in a similar way, but using polynomial inequalities

$$\|p(x_1, \ldots, x_n)\| \leqslant K_p \qquad (x_i \in A,\ \|x_i\| \leqslant 1 \ \ (1 \leqslant i \leqslant n))$$

instead of polynomial identities.

While the class IR is interesting, functional analysts are much more inclined to consider Banach algebras isomorphic to operator algebras by isomorphisms which are bicontinuous, but not necessarily isometric. We therefore introduced in [3] the concept of a *semivariety*. A semivariety is simply the closure of a variety under the operation of taking images under bicontinuous isomorphisms. Unfortunately, the resulting class will not, in general, be a variety,

2000 *Mathematics Subject Classifications* 46H05; 16R10.

because it will not be closed under infinite products. (If algebras A_n are in the semivariety by virtue of their isomorphisms ϕ_n with algebras of the underlying variety and the numbers $\|\phi_n\|$ are unbounded, then $\prod A_n$ might not be in the semivariety.)

On the other hand, one can easily think of examples of semivarieties which are also varieties; just take a set of polynomials $\mathcal{P}$ and define

$$\mathcal{V} = \{A : A \text{ is a Banach algebra and } p(x_1, \ldots, x_n) = 0 \quad (x_1, \ldots, x_n \in A,\ p \in \mathcal{P})\}.$$

We shall call such varieties *algebraically defined (AD)*.

In this paper we shall prove (Theorem 7.1) that algebraically defined varieties are the only varieties which are also semivarieties. However, the main purpose of the paper is the development of the theory of varieties of Banach algebras to the point where such questions can be answered easily. In particular, we shall formalize the notion of a 'graded normed algebra' and study examples of these which correspond to the algebraic notion of 'relatively free' algebras in a variety.

2 Varieties and semivarieties

Throughout, our Banach algebras will be over the complex field (this is unimportant — the real case is similar) and will not necessarily have identity elements; if they happen to have identity elements, these elements will not necessarily be of norm one. The latter stipulation is crucial: the theory of varieties of unital Banach algebras is very different.

Definition 2.1 If $\{A_\lambda\}_{\lambda\in\Lambda}$ is a family of Banach algebras, their *product* is the Banach algebra

$$\prod_\lambda A_\lambda = \{x = (x_\lambda)_{\lambda\in\Lambda} : \|x\| = \sup_\lambda \|x_\lambda\| < \infty\}.$$

Definition 2.2 A class $\mathcal{V}$ of Banach algebras is a *variety* if it is closed under taking: (a) closed subalgebras, (b) quotients by closed ideals, (c) products in the sense just described, and (d) images under isometric isomorphisms.

When we speak of a *polynomial*, we shall mean a polynomial over the complex field, in finitely many non-commuting indeterminates, without constant term. (Polynomials with constant term play an analogous rôle in the theory of varieties of unital Banach algebras.)

If $p(X_1, \ldots, X_n)$ is a polynomial and A is a Banach algebra, we denote by A_1 the closed unit ball of A and define

$$\|p\|_A = \sup\{\|p(x_1, \ldots, x_n)\| : x_1, \ldots, x_n \in A_1\}.$$

Defining

$$\|p\|_1 = \text{the sum of the moduli of the coefficients of } p$$

we have $\|p\|_A \leqslant \|p\|_1$ for all polynomials p and Banach algebras A.

The analogue of Birkhoff's Theorem for varieties of Banach algebras is the following.

Theorem 2.3 [**3**, theorem 2.3] *A class $\mathcal{V}$ of Banach algebras is a variety if and only if there is a non-negative real-valued function $p \mapsto K_p$ on the set of all polynomials such that*

$$\mathcal{V} = \{A : \|p\|_A \leqslant K_p \text{ for all polynomials } p\}.$$

Essentially, this says that varieties are defined by polynomial inequalities. The two extreme cases are worth noting. First, if $K_p = 0$ then saying $\|p\| \leqslant K_p$ is equivalent to saying $p \equiv 0$ in A. Secondly, if $K_p \geqslant \|p\|_1$, then the inequality $\|p\|_A \leqslant K_p$ is automatically satisfied, and therefore represents no constraint on the algebra A. Thus a class defined by any set of polynomial identities and inequalities can be portrayed as being defined by a function $p \mapsto K_p$.

In [4] we have given a generalization of the original Birkhoff Theorem to universal algebras with relations from which this Banach algebra version may be derived.

Definition 2.4 A class $\mathcal{S}$ of Banach algebras is called a *semivariety* if there is a variety $\mathcal{V}$ such that $\mathcal{S}$ is precisely the class of all Banach algebras bicontinuously isomorphic with algebras in $\mathcal{V}$. We write $\mathcal{S} = \hat{\mathcal{V}}$.

There is an obvious way to construct varieties which are also semivarieties.

Definition 2.5 If $\mathcal{V}$ is a variety defined by polynomial identities, that is, if

$$\mathcal{V} = \{A : A \text{ is a Banach algebra and } \|p\|_A = 0 \ \ (p \in \mathcal{P})\}$$

for some set $\mathcal{P}$ of polynomials, then $\mathcal{V}$ is said to be *algebraically defined*, (abbreviated AD).

It is clear that an AD-variety is closed under isomorphisms and is therefore a semivariety. We shall show the converse: that every variety which is also a semivariety is necessarily AD.

3 Graded algebras

To establish some notation, we recall that an associative algebra A is a *graded algebra* if

$$A = \bigoplus_{j=1}^{\infty} A^{(j)}$$

where the subspaces $A^{(j)}$ are such that

$$A^{(i)}A^{(j)} \subseteq A^{(i+j)} \qquad (1 \leqslant i, j < \infty).$$

For $1 \leqslant k < \infty$ we shall write

$$A^{(\leqslant k)} = \bigoplus_{j=1}^{k} A^{(j)}$$

and

$$A^{(>k)} = \bigoplus_{j=k+1}^{\infty} A^{(j)}.$$

We note that the latter is an ideal of A, so we can form the quotient algebra

$$A^{[k]} = A/A^{(>k)},$$

which, as a linear space, is canonically isomorphic with $A^{(\leqslant k)}$.

Definition 3.1 In every graded algebra A, there are natural projections $\pi_j : A \to A^{(j)}$ $(1 \leqslant j < \infty)$ such that $x = \sum \pi_j(x)$ $(x \in A)$. A *graded normed algebra* is a graded algebra which is also a normed algebra, such that all these projections are continuous.

Remark 3.2 It follows that the subspaces

$$\ker \pi_j = \bigoplus_{i \neq j} A^{(i)} \qquad (1 \leqslant j < \infty)$$

are all closed in A, and hence the subspaces $A^{(k)}$, $A^{(\leqslant k)}$ and $A^{(>k)}$ are all closed in A. In particular, since $A^{(>k)}$ is a closed ideal, $A^{[k]} = A/A^{(>k)}$ is a normed algebra.

Remark 3.3 There are algebras which are graded and normed but in which the projections π_j are not continuous. As an example, consider the subalgebra P of $C[0,1]$ (with the usual supremum norm) consisting of the polynomials, the subspaces $P^{(j)}$ being the sets of homogeneous polynomials of degree j for $j = 1, 2, 3, \ldots$.

On the other hand, the polynomials as a subalgebra of the disk algebra do form a graded normed algebra: the coefficients of a polynomial are bounded in modulus by the sup norm of the polynomial over the unit disk, and so the projections π_j onto the homogeneous components are continuous, of norm 1.

Definition 3.4 A *graded Banach algebra* is a Banach algebra which has a graded normed algebra as a dense subalgebra.

Remarks 3.5

1. The projections $\pi_j : A \to A^{(j)}$ of a graded normed algebra extend by continuity to projections $\pi_j : \tilde{A} \to \overline{A^{(j)}}$ from the completion $\tilde{A}$ onto the closures of the $A^{(j)}$ in $\tilde{A}$.

2. A graded Banach algebra is not itself a graded algebra, in general; indeed, a simple Baire Category Theorem argument shows that a Banach algebra cannot be a graded algebra unless $A^{(j)} = \{0\}$ for all but finitely many j.

The following proposition describes a situation we shall encounter later.

Proposition 3.6 *Suppose that A is a graded normed algebra, and k a positive integer, such that $A^{(>k)}$ is of finite codimension in A. Let $\tilde{A}$ be the completion of A and $\overline{A^{(>k)}}$ the closure of $A^{(>k)}$ in $\tilde{A}$. Then the natural mapping*

$$a + A^{(>k)} \mapsto a + \overline{A^{(>k)}} \ : \ A^{[k]} = A/A^{(>k)} \to \tilde{A}/\overline{A^{(>k)}}$$

is an isometric isomorphism (of Banach algebras).

Proof. This is an elementary result, the fact that the map is a linear isomorphism is a special case of the following easy lemma.

Lemma 3.7 *Let X be a normed space with completion $\tilde{X}$ and let Y be a closed subspace of X, of finite codimension, with closure $\overline{Y}$ in $\tilde{X}$. Then*

$$x + Y \mapsto x + \overline{Y} \ : \ X/Y \to \tilde{X}/\overline{Y}$$

is an isometric linear isomorphism.

The fact that $\overline{A^{(>k)}}$ is a closed ideal of $\tilde{A}$ follows from the fact that $A^{(>k)}$ is an ideal of A, and the linear isomorphism is then clearly an algebra isomorphism.

In the purely algebraic setting, if A is a graded algebra and I is a (two-sided) ideal of A, we write $I^{(j)} = \pi_j(I)$ and call I a *graded ideal* if $I^{(j)} \subseteq I$ for all j. In this situation, we have $I^{(j)} = I \cap A^{(j)}$ for all j, and the quotient algebra A/I is graded:

$$A/I = \bigoplus_{j=1}^{\infty} A^{(j)}/I^{(j)}.$$

If, now, we let A be a graded normed algebra and I a closed graded ideal, then $I^{(j)} = I \cap A^{(j)}$ is closed in $A^{(j)}$, so $A^{(j)}/I^{(j)}$ is a normed algebra, for each j. It is easy to check that A/I is a graded normed algebra; i.e. that the mappings

$$x + I \mapsto \pi_j(x) + I^{(j)} : A/I \to A^{(j)}/I^{(j)} \qquad (j = 1, 2, 3, \ldots)$$

are continuous.

4 Relatively free algebras

We now look at the algebraic concept of 'relatively free algebras in a variety' as it applies in varieties of Banach algebras.

Definition 4.1 Let F be the free algebra with generators $X_1, X_2, \ldots$. We make this an (incomplete) normed algebra by defining

$$\|p(X_1, \ldots, X_n)\| = \|p\|_1,$$

for all polynomials p.

The algebra F is graded:

$$F = \bigoplus F^{(j)}$$

where $F^{(j)}$ is the set of polynomials of degree j, and F is a graded normed algebra.

Definition 4.2 Given a class $\mathcal{C}$ of algebras, we consider the ideal $\mathcal{I}$ of F consisting of those p such that the identity $p(x_1, \ldots, x_n) = 0$ holds in every $A \in \mathcal{C}$. We call $\mathcal{I}$ the *ideal of identities* of $\mathcal{C}$.

It is a well-known (purely algebraic) fact that the polynomial identities holding in a class of algebras follow from the homogeneous polynomial identities holding in that class; that is, that $\mathcal{I}$ is a graded ideal. Furthermore, $\mathcal{I}$ is closed in the normed algebra F. This is particularly easy to see if the class consists of normed algebras, but it is true generally. Suppose (p_n) is a sequence of polynomials in $\mathcal{I}$ with $p_n \to p$. Then we may pass to homogeneous components: $p_n^{(j)} \to p^{(j)}$ for each j, since F is a graded normed algebra. Suppose $p^{(j)} = p^{(j)}(X_1, \ldots, X_N)$; i.e. observe that this $p^{(j)}$ involves only finitely many of the variables X_i. Let $q_{n,j}$ be the polynomial obtained from $p_n^{(j)}$ by setting $X_i = 0$ for all $i > N$. Then, the polynomials $q_{n,j}$ are identities for the class of algebras, so $q_{n,j} \in \mathcal{I}$. Moreover, by the definition of the norm $\|.\|_1$ on F, we have $q_{n,j} \to p^{(j)}$. However, we are now working inside the finite-dimensional space of polynomials homogeneous of degree j in $X_1, \ldots, X_N$ and $\mathcal{I}$ intersects this space in a linear subspace which is necessarily closed. Therefore $q_{n,j} \to p^{(j)}$ implies $p^{(j)} \in \mathcal{I}$. This holds for all j, up to the degree of p, so we have $p \in \mathcal{I}$.

It follows that we can form a graded normed algebra

$$R = F/\mathcal{I} = \bigoplus F^{(j)}/\mathcal{I}^{(j)}.$$

If $\mathcal{C}$ is a variety in the algebraic sense, then R is just the relatively free algebra on countably infinitely many generators.

Lemma 4.3 *In R we have $\|X_i + \mathcal{I}\| = 1$ for all i, (except in the trivial case $\mathcal{I} = F$).*

Proof. If $p \in \mathcal{I}$ and $p^{(j)}$ is its jth degree component, then we must have $p^{(1)}(X_1, X_2, \ldots, X_n) = 0$ for otherwise the class of algebras $\mathcal{C}$ would satisfy the identity $p^{(1)} \equiv 0$ and so would consist only of the trivial algebra $\{0\}$, whence $\mathcal{I} = F$. Then

$$\|X_i + p\|_1 = \|X_i\|_1 + \|p^{(2)}\|_1 + \|p^{(3)}\|_1 + \ldots \geqslant \|X_i\|_1 = 1,$$

so $\|X_i + \mathcal{I}\| \geqslant 1$. The reverse inequality is trivial.

The algebra R is characterised by the following universality property.

Theorem 4.4 *Let $\mathcal{I}$ be the ideal of identities of some class of algebras $\mathcal{C}$ and let B be an algebra satisfying the identities of $\mathcal{I}$. Then for every sequence $b_1, b_2, \ldots \in B$ there is a unique homomorphism $\theta : R \to B$ such that $\theta(X_i + \mathcal{I}) = b_i$ for all i.*

If, further, B is a normed algebra and the elements b_i are all of norm 1, then θ is continuous, with $\|\theta\| = 1$.

Proof. The universality of F implies that there is a homomorphism $\theta_0 : F \to B$ with $\theta_0(X_i) = b_i$ $(1 \leqslant i \leqslant n)$. In the normed case,

$$\|\theta_0(p(X_1, \ldots, X_n))\| = \|p(b_1, \ldots, b_n)\| \leqslant \|p\|_1 = \|p(X_1, \ldots, X_n)\|_F,$$

for every element $p(X_1, \ldots, X_n) \in F$, so θ_0 is continuous of norm 1.

Since $\theta_0(\mathcal{I}) = \{0\}$, the homomorphism θ_0 induces a homomorphism $\theta : R \to B$ as desired, and in the normed case $\|\theta\| \leqslant 1$. However, $\|X_i + \mathcal{I}\| = 1$ by the previous lemma, and $\|b_i\| = 1$, so $\|\theta\| = 1$.

The solution of our test question relies on our working with finite-dimensional algebras. We first observe that all the above can be done in a version with a finite set of generators $\{X_1, \ldots, X_n\}$, obtaining a free algebra F_n, an ideal $\mathcal{I}_n$ of n-variable identities of $\mathcal{C}$, and a quotient algebra

$$R_n = F_n/\mathcal{I}_n = \bigoplus F_n^{(j)}/\mathcal{I}_n^{(j)},$$

which is again a graded normed algebra. An n-variable version of Theorem 4.4 is provable by the same method.

To get a finite-dimensional algebra, we "cut off" the algebra R_n after the kth degree to obtain $R_n^{[k]} = R_n/R_n^{(>k)}$, which is a finite-dimensional algebra with a natural quotient algebra norm. (In fact, it is a graded normed algebra with $\left(R_n^{[k]}\right)^{(j)} = \{0\}$ for $j > k$.)

The norm in F_n has the special property that for $p \in F_n$ with homogeneous components $p^{(j)}$,

$$\|p\|_{F_n} = \sum_j \left\| p^{(j)} \right\|_{F_n^{(j)}}.$$

Consequently the norm in R_n has the same property. The following consequence of these observations will be needed later.

Lemma 4.5 *The linear space isomorphisms $F_n^{[k]} \cong F_n^{(\leqslant k)}$ and $R_n^{[k]} \cong R_n^{(\leqslant k)}$ are isometries.*

Not all graded normed algebras have this property.

5 Maximal algebras

Definition 5.1 For a class of Banach algebras $\mathcal{V}$ and a polynomial p, we define

$$\|p\|_{\mathcal{V}} = \sup\{\|p\|_A : A \in \mathcal{V}\}.$$

Remark 5.2 Theorem 2.3 says that $\mathcal{V}$ is a variety if and only if

$$\mathcal{V} = \{A : \|p\|_A \leqslant \|p\|_{\mathcal{V}}\}.$$

The supremum in the definition of $\|p\|_{\mathcal{V}}$ is attained, in the following very strong sense.

Lemma 5.3 *For a variety $\mathcal{V}$, there exists $M \in \mathcal{V}$ and a sequence $a_1, a_2, \ldots \in M_1$ (the unit ball of M) such that, for every polynomial $p(X_1, \ldots, X_n)$,*

$$\|p(a_1, a_2, \ldots, a_n)\|_M = \|p\|_{\mathcal{V}}. \tag{1}$$

Proof. For every polynomial $p(X_1, \ldots, X_n)$ and every $\varepsilon > 0$, we find $C(p, \varepsilon) \in \mathcal{V}$ such that

$$\|p\|_{C(p,\varepsilon)} > \|p\|_{\mathcal{V}} - \varepsilon$$

and $x_1, x_2, \ldots \in C(p, \varepsilon)_1$ such that

$$\|p(x_1, \ldots, x_n)\| > \|p\|_{C(p,\varepsilon)} - \varepsilon.$$

Then let M be the product of all the algebras $C(p, \varepsilon)$ and, for each i, let a_i be the element of M whose coordinates are the x_i, one for each pair (p, ε). The result follows.

The algebra M produced in this lemma is certainly not unique: any superalgebra of M would satisfy the requirements.

Definition 5.4 We replace M by the closure of the subalgebra M_0 generated by the a_i and call this the *maximal algebra* of the variety with *distinguished generators* $a_1, a_2, \ldots$. We call M_0 the *core* of the maximal algebra M

The maximal algebra M, together with its generators $a_1, a_2, \ldots$, and core M_0 is unique up to isomorphism because the condition (1) defines the norm on M_0 and hence defines M. Moreover, M is universal for the algebras in $\mathcal{V}$ in the following sense.

Theorem 5.5 *With the above notation, if $A \in \mathcal{V}$ and $x_1, x_2, \ldots \in A_1$, then there is a unique continuous homomorphism $\chi : M \to A$ with $\chi(a_i) = x_i$ $(i = 1, 2, \ldots)$. We have $\|\chi\| \leqslant 1$. If $\|x_i\| = 1$ for some i, then $\|\chi\| = 1$.*

Proof. We must define χ on M_0 by

$$\chi(p(a_1, a_2, \ldots)) = p(x_1, x_2, \ldots)$$

for all polynomials p. Equation (1) implies

$$\|p(a_1, a_2, \ldots, a_n)\|_M \geqslant \|p\|_A \geqslant \|p(x_1, x_2, \ldots, x_n)\|$$

for all p; hence χ is continuous on M_0 with $\|\chi\| \leqslant 1$ and the result follows.

We shall also need the finitely generated version of M.

Definition 5.6 We define $M_{n,0}$ to be the subalgebra of M_0 generated by $a_1, a_2, \ldots, a_n$ and M_n to be the closure of $M_{n,0}$ in M.

Theorem 5.5 has an obvious n-variable version giving a homomorphism $\chi : M_n \to A$ sending $a_1, a_2, \ldots, a_n$ to prescribed points $x_1, x_2, \ldots, x_n \in A_1$.

6 The Relationship between M and R

Suppose $\mathcal{V}$ is a variety of Banach algebras with maximal algebra M. By Theorem 4.4, there is an isomorphism $\theta : R \to M_0$ which is continuous, but not necessarily bicontinuous, such that $\theta(\overline{X_i}) = a_i$ for all i, where the elements $\overline{X_i} = X_i + \mathcal{I}$ are the generators of R and the a_i are the distinguished generators of M_0. This extends to a homomorphism $\tilde{R} \to M$.

If, further, $\mathcal{V}$ is an AD-variety, then R and $\tilde{R}$ satisfy the polynomial identities defining $\mathcal{V}$, and so $\tilde{R} \in \mathcal{V}$. Theorem 5.5 then gives a homomorphism $\chi : M \to \tilde{R}$ of norm at most one, with $\chi(a_i) = \overline{X_i}$ for all i, which is therefore inverse to θ. Therefore θ is an isometric isomorphism between R and M_0; (and $\tilde{R}$ and M are isometrically isomorphic). This discussion proves the following proposition.

Proposition 6.1 *Let $\mathcal{V}$ be a variety of Banach algebras. Then R with distinguished generators $\overline{X_i}$ $(i = 1, 2, \ldots)$ is algebraically isomorphic with the core M_0 of the maximal algebra of the variety with distinguished generators a_i.*

If, further, $\mathcal{V}$ is an AD-variety, then the completion $\tilde{R}$ of R with distinguished generators $\overline{X_i}$ is isometrically isomorphic with the maximal algebra M of the variety with distinguished generators a_i.

The corresponding results for the n-generator algebras R_n and $M_{n,0}$ follow immediately. This proposition allows us to transfer the graded structure of R to M_0.

Corollary 6.2 *The algebras M_0 and $M_{n,0}$ $(n = 1, 2, 3, \ldots)$ are graded normed algebras:*

$$M_0 = \bigoplus_j M_0^{(j)}, \qquad M_{n,0} = \bigoplus_j M_{n,0}^{(j)}, \tag{2}$$

where $M_0^{(j)}$, $(M_{n,0}^{(j)})$ are the subspaces spanned by the elements $p(a_1, \ldots)$, (respectively, $p(a_1, \ldots, a_n)$), for polynomials p homogeneous of degree j in the distinguished generators $a_1, a_2, \ldots$.

Further, $M_0 \cong R$ and $M_{n,0} \cong R_n$ as graded algebras.

Proof. The fact that M_0 and $M_{n,0}$ are graded follows from their algebraic isomorphisms with R and R_n, respectively, the grading described here being just the grading of R and R_n carried over by the isomorphisms to M_0 and $M_{n,0}$. The isomorphism is therefore a graded algebra isomorphism.

In any Banach algebra A, the equation

$$p^{(j)}(x_1,\ldots,x_n) = \frac{1}{2\pi i}\int_0^{2\pi} e^{-ij\theta} p\left(e^{i\theta}x_1,\ldots,e^{i\theta}x_n\right) d\theta \qquad (x_1,\ldots,x_n \in A)$$

yields

$$\|p^{(j)}\|_A \leqslant \|p\|_A.$$

Hence

$$\|p^{(j)}(a_1,\ldots,a_n)\|_M = \|p^{(j)}\|_{\mathcal{V}} \leqslant \|p\|_{\mathcal{V}} = \|p(a_1,\ldots,a_n)\|_M.$$

This not only proves again that (2) is a grading, but also shows that the maps

$$p(a_1,\ldots,a_n) \mapsto p^{(j)}(a_1,\ldots,a_n)$$

mapping $M_0 \to M_0^{(j)}$ (and $M_{n,0} \to M_{n,0}^{(j)}$) are continuous, so M_0 and $M_{n,0}$ $(n = 1, 2, 3, \ldots)$ are graded normed algebras.

Corollary 6.3 *The algebras $R_n^{[k]}$ and $M_{n,0}^{[k]}$ are algebraically isomorphic.*

In AD varieties, the construction of the maximal algebra as $\tilde{R}$ is clearly preferable to that involving a huge product of all the algebras $C(p,\varepsilon)$. However, for general (non-AD) varieties, the maximal algebra cannot be constructed as a quotient of a free Banach algebra on generators corresponding to the distinguished generators. It is a quotient of a free Banach on an uncountable set of generators — every Banach algebra is expressible in this way — but this is not helpful. Properties of general maximal algebras must be established from the original definition.

The following examples elucidate the nature of the maximal algebra of a non-AD-variety. (In these examples we have a clash of two, otherwise convenient, notations, In the following two examples, M_1, R_1, etc. will mean singly-generated algebras, i.e. M_n, R_n, etc. for $n = 1$, rather than the unit balls of M, R.)

Example 6.4 Consider the variety IQ of quotients of uniform algebras; the smallest variety containing the algebra $\mathbb{C}$. For a polynomial $p(X)$ in one variable, $\|p\|_{IQ} = \|p\|_{\mathbb{C}}$ is just the norm of p as an element of the disc algebra $A(D)$. Thus M_1 is canonically isometrically isomorphic to $A(D)$, with $M_{1,0}$ corresponding to the subalgebra of polynomials. On the other hand, $\mathcal{I} = \{0\}$, so $R_1 = F_1$ which is just an incomplete ℓ^1-sum of copies of $\mathbb{C}$. Thus $\tilde{R}_1$ is isometrically isomorphic to $(\ell^1, *)$, the algebra of absolutely convergent Fourier series.

Example 6.5 Let $w(j)$ $(j = 1, 2, 3, \ldots)$ be a decreasing sequence of positive numbers such that $w(1) = 1$,

$$w(j_1 + \ldots + j_k) \leqslant w(j_1)\ldots w(j_k) \qquad (1 \leqslant j_1, \ldots, j_k < \infty),$$

and $w(j)^{1/j} \to 0$ as $j \to \infty$. Let $\mathcal{V}$ be the variety defined by the polynomial inequalities

$$\|x_1 x_2 \ldots x_j\| \leqslant w(j) \qquad (x_1, x_2, \ldots, x_j \in A_1).$$

Then M_1 is the weighted sequence algebra $\left(\ell^1(w), *\right)$, a well-known radical Banach algebra, whilst $\tilde{R}_1$ is again the algebra $(\ell^1, *)$.

These example illustrate the distinction between $\tilde{R}$ and M for non-AD-varieties. Both are completed topological sums of spaces of homogeneous polynomials; the spaces $R_n^{(j)}$ and $M_n^{(j)}$ are finite-dimensional, so the canonical algebraic isomorphism between R and M_0 restricts to a bicontinuous isomorphism between them, but the norm of this isomorphism depends on j. This is illustrated clearly in Example 6.5. Further, whilst the norm of an element of the direct sum R is the sum of the norms of the homogeneous components, this is not necessarily the case in M_0; it is true in Example 6.5, but not in Example 6.4.

7 The 'Test Question'

Theorem 7.1 *If $\mathcal{V}$ is a variety and semivariety, then V is AD.*

In other words, a variety $\mathcal{V}$ is closed under taking images under bicontinuous (not necessarily isometric) isomorphisms if and only if $\mathcal{V}$ is AD.

Before we prove this, it is worth observing that it is possible for such a variety to be presented as a variety defined by polynomial inequalities rather than identities.

Example 7.2 The variety defined by the set of polynomial inequalities

$$\|p(x_1, \ldots, x_n)\| \leqslant \frac{1}{k} \qquad (x_1, \ldots, x_n \in A_1)$$

for $k = 1, 2, 3, \ldots$ is clearly just the AD-variety defined by the polynomial identity $p \equiv 0$.

This is trivial; a less obvious way of hiding polynomial identities is shown by the next example.

Example 7.3 The variety defined by the single polynomial inequality

$$\|x_0 + p(x_1, \ldots, x_n)\| \leqslant 1 \qquad (x_0, x_1, \ldots, x_n \in A_1)$$

is again just the AD-variety defined by the polynomial identity $p \equiv 0$.

If $\mathcal{V}$ is a (non-empty, proper) variety defined by finitely many homogeneous polynomial inequalities $\|p_i\|_A \leqslant K_i$ $(1 \leqslant i \leqslant M)$ with all the $K_i > 0$, then it is easy to show that $\mathcal{V}$ is not closed under isomorphisms. To see this, we first assume that $\deg p_i > 1$ for all i, since inequalities on degree 1 polynomials are either redundant (if $K_i \geqslant \|p_i\|_1$) or force the variety to be trivial (if $K_i < \|p_i\|_1$). We then take any algebra $A \notin \mathcal{V}$ and consider the algebra B consisting of the algebra A with $\|x\|_B = \lambda \|x\|_A$, for some constant $\lambda > 1$. For each i, we have $\|p_i\|_B \leqslant K_i$ when λ is sufficiently large. Therefore, for sufficiently large λ we have $B \in \mathcal{V}$. Since B is clearly isomorphic to A, this shows that $\mathcal{V}$ is not closed under isomorphisms.

This simple argument proves our theorem for varieties defined by finitely many homogeneous polynomial inequalities, but neither of our two examples above is of this type. The general proof depends on the graded algebra machinery we have set up earlier in the paper.

Proof. We adopt the notations introduced above. Let $\mathcal{I}$ be the ideal of identities of $\mathcal{V}$. We have to show that if B is a Banach algebra satisfying these identities, then $B \in \mathcal{V}$.

Given a polynomial $p(X_1, \ldots, X_n)$ and elements $b_1, \ldots, b_n \in B$ with $\|b_i\| = 1$ $(1 \leqslant i \leqslant n)$, we must show that

$$\|p(b_1, \ldots, b_n)\|_B \leqslant \|p\|_{\mathcal{V}}.$$

By the n-variable version of Theorem 4.4, there is a (unique) homomorphism of norm 1 from R_n into B mapping $\overline{X_i} \mapsto b_i$ ($1 \leqslant i \leqslant n$). Hence it now suffices to show that

$$\|p(\overline{X_1}, \ldots, \overline{X_n})\|_{R_n} \leqslant \|p\|_{\mathcal{V}}.$$

Let $k = \deg p$. Then the left hand side can be replaced by the norm in $R_n^{(\leqslant k)}$. Therefore, using Lemma 4.5, we need

$$\|p(\overline{X_1}, \ldots, \overline{X_n})\|_{R_n^{[k]}} \leqslant \|p\|_{\mathcal{V}}.$$

This will follow if $R_n^{[k]} \in \mathcal{V}$.

By Corollary 6.3 and Proposition 3.6, we have algebraic isomorphisms

$$R_n^{[k]} \cong M_{n,0}^{[k]} \cong M_n / \overline{M_{n,0}^{(>k)}}.$$

Since $R_n^{[k]}$ and $M_n/\overline{M_{n,0}^{(>k)}}$ are finite-dimensional, they are isomorphic as Banach algebras; but $M \in \mathcal{V}$, so $M_n \in \mathcal{V}$, so $M_n/\overline{M_{n,0}^{(>k)}} \in \mathcal{V}$ and $\mathcal{V}$ is closed under Banach algebra isomorphisms, so $R_n^{[k]} \in \mathcal{V}$ and the result follows.

References

[1] A. Bernard, Quotients of operator algebras. In *Seminar on uniform algebras* (University of Aberdeen, 1973).

[2] G. Birkhoff, On the structure of abstract algebras. *Proc. Camb. Philos. Soc.* **31** (1935), 433–454.

[3] P. G. Dixon, Varieties of Banach algebras. *Quart. J. Math. Oxford (2)* **27** (1976), 481–487.

[4] P. G. Dixon, Classes of algebraic systems defined by universal Horn sentences. *Algebra Universalis* **7** (1977), 315–339.

[5] C. Le Merdy, Representation of a quotient of a subalgebra of B(X). *Math. Proc. Camb. Philos. Soc.* **119** (1996), 83–90.

Department of Pure Mathematics,
University of Sheffield,
SHEFFIELD, S3 7RH,
England.

e-mail: P.Dixon@sheffield.ac.uk

WWW: http://www.shef.ac.uk/~pm1pgd/