Graded Banach algebras and varieties of Banach algebras
P. G. Dixon

1. Varieties: Algebraic Motivation.
2. Varieties of Banach Algebras: the Birkhoff Theorem; our 'Test Question'.
3. Graded Algebras: R, M and the relation between them.
4. The 'Test Question' answered.
5. Other directions.

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Algebraic Motivation

Consider linear associative algebras over the complex field.

Definition 1 A *variety* of algebras is a (non-empty) class of algebras closed under subalgebras, quotients, products and isomorphic images.

Definition 2 Let $p(X_1, \ldots, X_n)$ be a polynomial (unless otherwise specified, this will mean a polynomial in non-commuting indeterminates, without constant term). We say that an algebra A *obeys the law* $p = 0$ iff

$$p(x_1, \ldots, x_n) = 0 \qquad (x_1, \ldots, x_n \in A).$$

Example 1 An algebra "obeys the law $XY - YX = 0$" iff it is commutative.

Theorem 1 (Birkhoff's Theorem) *For a class $\mathcal{V}$ of algebras the following are equivalent:*

1. *$\mathcal{V}$ is a variety (closed under subalgebras, quotients, products and isomorphic images);*
2. *For some set P of polynomials*

$$\mathcal{V} = \{A : A \textit{ obeys the law } p = 0 \quad \forall p \in P\}.$$

Varieties of Banach algebras

Definition 3 If $\{A_\lambda\}_{\lambda\in\Lambda}$ is a family of Banach algebras, their *product* is the Banach algebra

$$\prod_\lambda A_\lambda = \{x = (x_\lambda)_{\lambda\in\Lambda} : \|x\| = \sup_\lambda \|x_\lambda\| < \infty\}.$$

Definition 4 A (non-empty) class $\mathcal{V}$ of Banach algebras is a *variety* if it is closed under taking: (a) closed subalgebras, (b) quotients by closed ideals, (c) products in the sense just described, and (d) images under isometric isomorphisms.

When we speak of a *polynomial*, we shall mean a polynomial over the complex field, in finitely many non-commuting indeterminates, without constant term. (Polynomials with constant term play an analogous rôle in the theory of varieties of unital Banach algebras.)

If $p(X_1, \ldots, X_n)$ is a polynomial and A is a Banach algebra, we denote by A_1 the closed unit ball of A and define

$$\|p\|_A = \sup\{\|p(x_1, \ldots, x_n)\| : x_1, \ldots, x_n \in A_1\}.$$

Defining

$$\|p\|_1 = \Sigma(\text{moduli of the coefficients of } p),$$

we have $\|p\|_A \leqslant \|p\|_1$ for all polynomials p and Banach algebras A.

Theorem 2 (Birkhoff for Banach algebras) *A class $\mathcal{V}$ of Banach algebras is a variety if and only if there is a non-negative real-valued function $p \mapsto K_p$ on the set of all polynomials such that*

$$\mathcal{V} = \{A : \|p\|_A \leqslant K_p \quad \forall \text{ polynomials } p\}.$$

Essentially, this says that varieties are defined by polynomial inequalities. The two extreme cases are worth noting. First, if $K_p = 0$ then saying $\|p\| \leqslant K_p$ is equivalent to saying $p \equiv 0$ in A. Secondly, if $K_p \geqslant \|p\|_1$, then the inequality $\|p\|_A \leqslant K_p$ is automatically satisfied, and therefore represents no constraint on the algebra A. Thus a class defined by any set of polynomial identities and inequalities can be portrayed as being defined by a function $p \mapsto K_p$.

In [3] we have given a generalization of the original Birkhoff Theorem to universal algebras with relations from which this Banach algebra version may be derived.

Remark 1 For a class of Banach algebras $\mathcal{V}$ and a polynomial p, if

$$\|p\|_{\mathcal{V}} := \sup\{\|p\|_A : A \in \mathcal{V}\}.$$

then Theorem 2 says that $\mathcal{V}$ is a variety iff

$$\mathcal{V} = \{A : \|p\|_A \leqslant \|p\|_{\mathcal{V}}\},$$

Example 1: Q-algebras

Definition 5 A Banach algebra A is a **uniform algebra** it is a closed subalgebra of $C(X)$ for some X.

Definition 6 (Q-algebras: J. Wermer, [13]) A Banach algebra A is a **Q-algebra (IQ-algebra)** if it is isomorphic (isometrically isomorphic) to the quotient of a uniform algebra by a closed ideal.

Example 2 The class IQ of all IQ-algebras is a variety. In fact, it is the smallest variety containing $\mathbb{C}$: the **variety generated by** $\mathbb{C}$. It is therefore defined by the polynomial norm inequalities

$$\|p\|_A \leqslant \|p\|_{\mathbb{C}} = \sup\{|p(z_1, \ldots, z_n)| : |z_i| \leqslant 1\}.$$

This is just Craw's Lemma for IQ-algebras.

Example 2: Operator algebras

Example 3 Let R, IR be the classes of algebras isomorphic (isometrically isomorphic) to closed subalgebras of $\mathcal{B}(H)$ for H a (complex) Hilbert space ("operator algebras").

(WARNING: these subalgebras are not necessarily self-adjoint.)

Theorem 3 (Bernard) *The class IR is closed under quotients.*

It is then easy to see that IR is a variety: the variety generated by $\mathcal{B}(H)$, where H is separable Hilbert space.

Corollary 1 *A Banach algebra A is isometrically isomorphic to an operator algebra iff*

$$\|p\|_A \leqslant \|p\|_{\mathcal{B}(H)}$$

for all polynomials p.

Remark 2 Since

$$\mathbb{C} \cong \{\lambda I \in \mathcal{B}(H) : \lambda \in \mathbb{C}\},$$

$\mathbb{C}$ is an IR-algebra, so we have

$$\|p\|_{IR} \geqslant \|p\|_{\mathbb{C}} = \|p\|_{IQ};$$

$$IQ \subseteq IR.$$

This inclusion is proper, since IR contains non-commutative algebras, but, more subtlely, there are commutative IR algebras which are not Q-algebras (Varopoulos).

Proof of Theorem 2 The 'if' part is straightforward. We prove the 'only if'.

For every polynomial p and every $\varepsilon > 0$, we find $C(p, \varepsilon) \in \mathcal{C}$ such that

$$\|p\|_{C(p,\varepsilon)} > \|p\|_{\mathcal{C}} - \varepsilon.$$

If M is the product of all the algebras $C(p, \varepsilon)$, then $M \in \mathcal{C}$ and

$$\|p\|_M = \|p\|_{\mathcal{C}}$$

for every polynomial p.

It is clear that

$$\mathcal{C} \subseteq \{A : \|p\|_A \leqslant \|p\|_{\mathcal{C}} \text{ for all } p\}.$$

We prove the reverse inclusion. Suppose A is a Banach algebra such that, for all polynomials p,

$$\|p\|_A \leqslant \|p\|_{\mathcal{C}} = \|p\|_M.$$

Put

$$X = \Big(M_1\Big)^{A_1},$$

the set of all functions from A_1 into M_1.

Let $\Gamma = M^X$, meaning the algebra of all bounded functions from X into M, this is the Banach algebra product of X copies of M, so $\Gamma \in \mathcal{C}$. For $a \in A_1$, define $\gamma_a \in \Gamma$ by

$$\gamma_a(x) = x(a) \qquad (x \in X).$$

Then $\|\gamma_a\| = 1$. Let

$$U_0 = \text{subalgebra of } \Gamma \text{ generated by } \{\gamma_a : a \in A_1\},$$

and define a homomorphism

$$\theta : U_0 \to A \qquad \theta(\gamma_a) = a \qquad (a \in A_1).$$

A typical element of U_0 is

$$u = p(\gamma_{a_1}, \ldots, \gamma_{a_n})$$

for some polynomial p and distinct $a_1, \ldots, a_n \in A_1$. Then

$$\theta(u) = p(a_1, \ldots, a_n).$$

We show that θ is well-defined ($u = 0 \Rightarrow \theta(u) = 0$) and continuous with $\|\theta\| \leqslant 1$:

$$\begin{aligned}
\|\theta(u)\| &= \|p(a_1, \ldots, a_n)\| \\
&\leqslant \|p\|_A \\
&\leqslant \|p\|_M \\
&= \sup\{\|p(x(a_1), \ldots, x(a_n))\| : x \in X\} \\
&= \sup\{\|u(x)\| : x \in X\} \\
&= \|u\|.
\end{aligned}$$

Let $U = \overline{U_0}$. Then $\theta : U \to A$ is surjective, since $\theta : U_0 \to A$ is already surjective. To show that

$$A \cong U/\ker\theta, \text{ isometrically,}$$

we observe that for $a \in A$ with $\|a\| = 1$ we have $a = \theta(\gamma_a)$ and $\|\gamma_a\| = 1$. This, with $\|\theta\| \leqslant 1$ proves the natural isomorphism to be isometric.

We have that the closure of $\mathcal{C}$ under

$$\begin{aligned}
\text{products} &\quad \Rightarrow \quad \Gamma \in \mathcal{C} \\
\text{closed subalgebras} &\quad \Rightarrow \quad U \in \mathcal{C} \\
\text{quotients} &\quad \Rightarrow \quad U/\ker\theta \in \mathcal{C} \\
\text{isometric isomorphisms} &\quad \Rightarrow \quad A \in \mathcal{C},
\end{aligned}$$

as required.

Corollary 2 *Every variety of Banach algebras is singly generated.*

The algebra M in the above proof is a generator of the variety $\mathcal{C}$.

Semivarieties

To accommodate Q and R, the classes of Banach algebras isomorphic (i.e. bicontinuously isomorphic) to some IQ-algebra, IR-algebra, respectively, we introduce the notion of a "semivariety".

Definition 7 A **semivariety** is a class $\mathcal{S}$ of Banach algebras such that there is a variety $\mathcal{V}$ of Banach algebras such that $\mathcal{S}$ is the class of all Banach algebras (bicontinuously) isomorphic to some algebra in $\mathcal{V}$.

Theorem 4 (PGD, [2]) *For a non-empty class $\mathcal{S}$ of Banach algebras, the following are equivalent:*

(i) $\mathcal{S}$ is a semivariety of Banach algebras;

(ii) there is a function $p \mapsto K(p)$ from the set of all polynomials to the non-negative reals such that $\mathcal{S}$ is precisely the class of Banach algebras A for which there exist $M, \delta > 0$ such that, for every polynomial $p(X_1, \ldots, X_n)$ and all $x_1, \ldots, x_n \in A_\delta$ $(= \{x \in A : \|x\| \leqslant \delta\})$,

$$\|p(x_1, \ldots, x_n)\| \leqslant M\, K(p);$$

(iii) ditto, but for homogeneous polynomials only.

Examples of Semivarieties

1. The semivariety of all Q-algebras is the class of Banach algebras A for which there exist $M, \delta > 0$ such that, for every polynomial p,

$$\|p(x_1, \ldots, x_n)\| \leqslant M\|p\|_{\mathbb{C}} \qquad (x_1, \ldots, x_n \in A_\delta).$$

2. The semivariety of all R-algebras is the class of Banach algebras A for which there exist $M, \delta > 0$ such that, for every polynomial p,

$$\|p(x_1, \ldots, x_n)\| \leqslant M\|p\|_{\mathcal{B}(H)} \quad (x_1, \ldots, x_n \in A_\delta).$$

The Q-algebras form a proper subset of the commutative R-algebras (Varopoulos [12] — a very deep result).

Algebraically-defined varieties

Most varieties are not semivarieties — i.e. they are not closed under isomorphisms. However, there is an obvious way to construct some which are.

Definition 8 If $\mathcal{V}$ is a variety defined by polynomial identities, that is, if

$$\mathcal{V} = \{A : A \text{ is a Banach alg. } \& \ \|p\|_A = 0 \ (p \in \mathcal{P})\}$$

for some set $\mathcal{P}$ of polynomials, then $\mathcal{V}$ is said to be *algebraically defined*, (abbreviated AD).

It is clear that an AD-variety is closed under isomorphisms and is therefore a semivariety. Thus AD $\Rightarrow$ closed under isomorphism.

Question 1 [our 'Test question'] Is it true that for varieties

$$\text{closed under isomorphism} \quad \Rightarrow \quad \text{AD?}$$

i.e. if $\mathcal{V}$ is a variety and a semivariety, is it necessarily AD?

This is a good 'test question' to test, and develop, our understanding of the theory.

Varieties which are Semivarieties

We shall see that the answer to our Test Question is positive, but it appears to involve some key constructions for the theory, so although it looks very elementary, it is a good question to ask to prompt further investigations.

To see that it is not completely elementary, consider the following examples where AD-varieties are defined by non-identity polynomial inequalities.

Examples 4

1. The inequalities

$$\|Np(X_1, \ldots, X_n)\|_A \leqslant 1 \qquad (\|X_i\| \leqslant 1)$$

for $N = 1, 2, 3, \ldots$ are collectively equivalent to the identity

$$p(X_1, ..., X_n) = 0.$$

2. Less trivially, the single inequality

$$\|X_0 + p(X_1, \ldots, X_n)\|_A \leqslant 1 \qquad (\|X_i\| \leqslant 1)$$

is equivalent to the identity

$$p(X_1, ..., X_n) = 0.$$

Proof. The fact that the polynomial identity implies the law is obvious. Suppose the law holds and apply it with

$$x_0 = \|p(x_1, \ldots, x_n)\|^{-1} p(x_1, \ldots, x_n).$$

We obtain, for $\|x_1\|, \ldots, \|x_n\| \leqslant 1$,

$$1 \geqslant \|(1 + \|p(x_1, \ldots, x_n)\|^{-1}) p(x_1, \ldots, x_n)\| = \|p(x_1, \ldots, x_n)\| + 1,$$

so $p(x_1, \ldots, x_n) = 0$. For general $x_1, \ldots, x_n$, we observe that the polynomial function

$$\mathbb{C} \to A : \lambda \mapsto p(\lambda x_1, \ldots, \lambda x_n)$$

vanishes for all sufficiently small λ. Therefore, by analytic continuation, it vanishes for all λ.$\Diamond$

It is fairly easy to show that if $\mathcal{V}$ is defined, as a variety, by inequalities on finitely many homogeneous polynomials, and $\mathcal{V}$ is also a variety, then $\mathcal{V}$ is AD. However, these hypotheses carefully avoid addressing the problems raised by the two examples above.

Graded algebras

An associative algebra A is a *graded algebra* if

$$A = \bigoplus_{j=1}^{\infty} A^{(j)}$$

where the subspaces $A^{(j)}$ are such that

$$A^{(i)} A^{(j)} \subseteq A^{(i+j)} \qquad (1 \leqslant i, j < \infty).$$

For $1 \leqslant k < \infty$ we write

$$A^{(\leqslant k)} = \bigoplus_{j=1}^{k} A^{(j)}, \qquad A^{(>k)} = \bigoplus_{j=k+1}^{\infty} A^{(j)}.$$

We note that the latter is an ideal of A, so we can form the quotient algebra

$$A^{[k]} = A / A^{(>k)},$$

which, as a linear space, is canonically isomorphic with $A^{(\leqslant k)}$.

Definition 9 In every graded algebra A, there are natural projections

$$\pi_j : A \to A^{(j)} \qquad (1 \leqslant j < \infty)$$

such that $x = \sum \pi_j(x)$ $(x \in A)$. A *graded normed algebra* is a graded algebra which is also a normed algebra, such that all these projections are continuous.

Remark 3 In a graded normed algebra, the subspaces

$$\ker \pi_j = \bigoplus_{i \neq j} A^{(i)} \qquad (1 \leqslant j < \infty)$$

are all closed in A, and hence the subspaces $A^{(k)}$, $A^{(\leqslant k)}$ and $A^{(>k)}$ are all closed in A. So $A^{[k]} = A/A^{(>k)}$ is a normed algebra.

Definition 10 A *graded Banach algebra* is a Banach algebra which has a graded normed algebra as a dense subalgebra.

Remark 4 There are algebras which are graded and normed but in which the projections π_j are not continuous. As an example, consider the subalgebra P of $C[0,1]$ (with the usual supremum norm) consisting of the polynomials, the subspaces $P^{(j)}$ being the sets of homogeneous polynomials of degree j for $j = 1, 2, 3, \ldots$.

On the other hand, the polynomials as a subalgebra of the disk algebra do form a graded normed algebra: the coefficients of a polynomial are bounded in modulus by the sup norm of the polynomial over the unit disk, and so the projections π_j onto the homogeneous components are continuous, of norm 1.

Remark 5 The projections $\pi_j : A \to A^{(j)}$ of a graded normed algebra extend by continuity to projections $\pi_j : \tilde{A} \to \overline{A^{(j)}}$ where
$\tilde{A} =$ completion of A and
$\overline{A^{(j)}} =$ closure of the $A^{(j)}$ in $\tilde{A}$.

Remark 6 A graded Banach algebra A is not itself a graded algebra, in general;
(Baire Category Theorem $\Rightarrow$ $A^{(j)} = \{0\}$ for all but finitely many j).

The following proposition describes a situation we shall encounter later.

Proposition 1 *Suppose that A is a graded normed algebra, and k a positive integer, such that $A^{(>k)}$ is of finite codimension in A. Let $\overline{A^{(>k)}}$ = closure of $A^{(>k)}$ in $\tilde{A}$. Then the natural mapping*

$$a + A^{(>k)} \mapsto a + \overline{A^{(>k)}} \ : \ A^{[k]} = A/A^{(>k)} \to \tilde{A}/\overline{A^{(>k)}}$$

is an isometric isomorphism (of Banach algebras).

Graded Ideals

In the purely algebraic setting, if A is a graded algebra and I is a (two-sided) ideal of A, we write $I^{(j)} = \pi_j(I)$ and call I a *graded ideal* if $I^{(j)} \subseteq I$ for all j.

In this situation, we have $I^{(j)} = I \cap A^{(j)}$ for all j, and the quotient algebra A/I is graded:

$$A/I = \bigoplus_{j=1}^{\infty} A^{(j)}/I^{(j)}.$$

Let A be a graded normed algebra and I a closed graded ideal. Then $I^{(j)} = I \cap A^{(j)}$ is closed in $A^{(j)}$, so $A^{(j)}/I^{(j)}$ is a normed algebra, for each j. It is easy to check that A/I is a graded normed algebra; i.e. that the mappings

$$x + I \mapsto \pi_j(x) + I^{(j)} : A/I \to A^{(j)}/I^{(j)}$$

$(j = 1, 2, 3, \ldots)$ are continuous.

Relatively free algebras

Definition 11 Let F be the free algebra with generators $X_1, X_2, \ldots$. We make this an (incomplete) normed algebra by defining

$$\|p(X_1, \ldots, X_n)\| = \|p\|_1,$$

for all polynomials p.

The algebra F is graded:

$$F = \bigoplus F^{(j)}$$

where $F^{(j)}$ is the set of (non-commutative) homogeneous polynomials of degree j

Then F is a graded normed algebra.

Definition 12 Given a class $\mathcal{V}$ of algebras, we consider the (closed, graded) ideal $\mathcal{I}$ of F consisting of those p such that the identity $p(x_1, \ldots, x_n) = 0$ holds in every $A \in \mathcal{V}$. We call $\mathcal{I}$ the *ideal of identities* of $\mathcal{V}$.

We can form a graded normed algebra

$$R = F/\mathcal{I} = \bigoplus F^{(j)}/\mathcal{I}^{(j)}.$$

If $\mathcal{C}$ is a variety in the algebraic sense, then R is just the relatively free algebra on countably infinitely many generators.

The algebra R is characterised by the following universality property.

Theorem 5 (Universality of R) *Let $\mathcal{I}$ be the ideal of identities of some class of algebras $\mathcal{V}$ and let B be an algebra satisfying the identities of $\mathcal{I}$. Then for every sequence $b_1, b_2, \ldots \in B$ there is a unique homomorphism $\theta : R \to B$ such that $\theta(X_i + \mathcal{I}) = b_i$ for all i.*

If, further, B is a normed algebra and the elements b_i are all of norm 1, then θ is continuous, with $\|\theta\| = 1$.

The solution of our test question relies on our working with finite-dimensional algebras. We first observe that all the above can be done in a version with a finite set of generators $\{X_1, \ldots, X_n\}$, obtaining a free algebra F_n, an ideal $\mathcal{I}_n$ of n-variable identities of $\mathcal{V}$, and a quotient algebra

$$R_n = F_n/\mathcal{I}_n = \bigoplus F_n^{(j)}/\mathcal{I}_n^{(j)},$$

which is again a graded normed algebra. An n-variable version of Theorem 5 is provable by the same method.

We "cut off" the algebra R_n after the kth degree:

$$R_n^{[k]} = R_n/R_n^{(>k)}.$$

This is a finite-dimensional algebra with the natural quotient algebra norm. (In fact, it is a graded normed algebra with $\left(R_n^{[k]}\right)^{(j)} = \{0\}$ for $j > k$.)

The norm in F_n has the special property that for $p \in F_n$ with homogeneous components $p^{(j)}$,

$$\|p\|_{F_n} = \sum_j \left\|p^{(j)}\right\|_{F_n^{(j)}}.$$

Consequently the norm in R_n has the same property. The following consequence of these observations will be needed later.

Lemma 1 *The linear space isomorphisms $F_n^{[k]} \cong F_n^{(\leqslant k)}$ and $R_n^{[k]} \cong R_n^{(\leqslant k)}$ are isometries.*

Not all graded normed algebras have this property.

Maximal algebras

Recall that

$$\|p\|_{\mathcal{V}} := \sup\{\|p\|_A : A \in \mathcal{V}\}.$$

The supremum this definition is attained, in the following very strong sense.

Lemma 2 *For a variety $\mathcal{V}$, there exists $M \in \mathcal{V}$ and a sequence $a_1, a_2, \ldots \in M_1$ (unit ball of M) such that, for every polynomial $p(X_1, \ldots, X_n)$,*

$$\|p(a_1, a_2, \ldots, a_n)\|_M = \|p\|_{\mathcal{V}}. \tag{1}$$

Proof. For every polynomial $p(X_1, \ldots, X_n)$ and every $\varepsilon > 0$, we find $C(p, \varepsilon) \in \mathcal{V}$ such that

$$\|p\|_{C(p,\varepsilon)} > \|p\|_{\mathcal{V}} - \varepsilon$$

and $x_1, x_2, \ldots \in C(p, \varepsilon)_1$ such that

$$\|p(x_1, \ldots, x_n)\| > \|p\|_{C(p,\varepsilon)} - \varepsilon.$$

Then let M be the product of all the algebras $C(p, \varepsilon)$ and, for each i, let a_i be the element of M whose coordinates are the x_i, one for each pair (p, ε). The result follows. $\Diamond$

The algebra M produced in this lemma is certainly not unique: any superalgebra of M would satisfy the requirements.

Definition 13 We replace M by the closure of the subalgebra M_0 generated by the a_i and call this the *maximal algebra* of the variety with *distinguished generators* $a_1, a_2, \ldots$. We call M_0 the *core* of the maximal algebra M

The maximal algebra M, together with its generators $a_1, a_2, \ldots$, and core M_0 is unique up to isomorphism because the condition (1) defines the norm on M_0 and hence defines M. Moreover, M is universal for the algebras in $\mathcal{V}$ in the following sense.

Theorem 6 (Universality of M) *With the above notation, if $A \in \mathcal{V}$ and $x_1, x_2, \ldots \in A_1$, then there is a unique continuous homomorphism $\chi : M \to A$ with $\chi(a_i) = x_i$ $(i = 1, 2, \ldots)$. We have $\|\chi\| \leqslant 1$. If $\|x_i\| = 1$ for some i, then $\|\chi\| = 1$.*

Theorem 6 has an obvious n-variable version giving a homomorphism $\chi : M_n \to A$ sending $a_1, a_2, \ldots, a_n$ to prescribed points $x_1, x_2, \ldots, x_n \in A_1$.

M and R

Suppose $\mathcal{V}$ is a variety of Banach algebras with maximal algebra M. By Theorem 5, there is an isomorphism $\theta : R \to M_0$ which is continuous, but not necessarily bicontinuous, such that $\theta(\overline{X_i}) = a_i$ for all i, where the elements $\overline{X_i} = X_i + \mathcal{I}$ are the generators of R and the a_i are the distinguished generators of M_0. This extends to a homomorphism $\tilde{R} \to M$.

If, further, $\mathcal{V}$ is an AD-variety, then R and $\tilde{R}$ satisfy the polynomial identities defining $\mathcal{V}$, and so $\tilde{R} \in \mathcal{V}$. Theorem 6 then gives a homomorphism $\chi : M \to \tilde{R}$ of norm at most one, with $\chi(a_i) = \overline{X_i}$ for all i, which is therefore inverse to θ. Therefore θ is an isometric isomorphism between R and M_0; (and $\tilde{R}$ and M are isometrically isomorphic). This discussion proves the following proposition.

Proposition 2 *Let $\mathcal{V}$ be a variety of Banach algebras. Then R with distinguished generators $\overline{X_i}$ $(i = 1, 2, \ldots)$ is algebraically isomorphic with the core M_0 of the maximal algebra of the variety with distinguished generators a_i.*

If, further, $\mathcal{V}$ is an AD-variety, then the completion $\tilde{R}$ of R with distinguished generators $\overline{X_i}$ is isometrically isomorphic with the maximal algebra M of the variety with distinguished generators a_i.

The corresponding results for the n-generator algebras R_n and $M_{n,0}$ follow immediately. This proposition allows us to transfer the graded structure of R to M_0.

Copying the grading:$R \to M$

Corollary 3 *The algebras M_0 and $M_{n,0}$ $(n = 1, 2, 3, \ldots)$ are graded normed algebras:*

$$M_0 = \bigoplus_j M_0^{(j)}, \qquad M_{n,0} = \bigoplus_j M_{n,0}^{(j)}, \tag{2}$$

where $M_0^{(j)}$, $(M_{n,0}^{(j)})$ are the subspaces spanned by the elements $p(a_1, \ldots)$, (respectively, $p(a_1, \ldots, a_n)$), for polynomials p homogeneous of degree j in the distinguished generators $a_1, a_2, \ldots$.

Further, $M_0 \cong R$ and $M_{n,0} \cong R_n$ as graded algebras.

Proof. The fact that M_0 and $M_{n,0}$ are graded follows from their algebraic isomorphisms with R and R_n, respectively, the grading described here being just the grading of R and R_n carried over by the isomorphisms to M_0 and $M_{n,0}$. The isomorphism is therefore a graded algebra isomorphism.

In any Banach algebra A, the equation

$$p^{(j)}(x_1, \ldots, x_n) = \frac{1}{2\pi i} \int_0^{2\pi} e^{-ij\theta} p\left(e^{i\theta} x_1, \ldots, e^{i\theta} x_n\right) d\theta \qquad (x_1, \ldots, x_n \in A)$$

yields

$$\|p^{(j)}\|_A \leqslant \|p\|_A.$$

Hence

$$\|p^{(j)}(a_1, \ldots, a_n)\|_M = \|p^{(j)}\|_{\mathcal{V}} \leqslant \|p\|_{\mathcal{V}} = \|p(a_1, \ldots, a_n)\|_M.$$

This not only proves again that (2) is a grading, but also shows that the maps

$$p(a_1, \ldots, a_n) \mapsto p^{(j)}(a_1, \ldots, a_n)$$

mapping $M_0 \to M_0^{(j)}$ (and $M_{n,0} \to M_{n,0}^{(j)}$) are continuous, so M_0 and $M_{n,0}$ $(n = 1, 2, 3, \ldots)$ are graded normed algebras. $\Diamond$

Corollary 4 *The algebras $R_n^{[k]}$ and $M_{n,0}^{[k]}$ are algebraically isomorphic.*

The 'Test Question'

Theorem 7 *If $\mathcal{V}$ is a variety closed under isomorphisms, then V is AD.*

Proof We adopt the notations introduced above. Let $\mathcal{I}$ be the ideal of identities of $\mathcal{V}$. We have to show that if B is a Banach algebra satisfying these identities, then $B \in \mathcal{V}$.

Given a polynomial $p(X_1, \ldots, X_n)$ and elements $b_1, \ldots, b_n \in B$ with $\|b_i\| = 1$ $(1 \leqslant i \leqslant n)$, we must show that

$$\|p(b_1, \ldots, b_n)\|_B \leqslant \|p\|_{\mathcal{V}}.$$

By the n-variable version of Theorem 5, there is a (unique) homomorphism of norm 1 from R_n into B mapping $\overline{X_i} \mapsto b_i$ $(1 \leqslant i \leqslant n)$. Hence it now suffices to show that

$$\|p(\overline{X_1}, \ldots, \overline{X_n})\|_{R_n} \leqslant \|p\|_{\mathcal{V}}.$$

Let $k = \deg p$. Then the left hand side can be replaced by the norm in $R_n^{(\leqslant k)}$. Therefore, using Lemma 1, we need

$$\|p(\overline{X_1}, \ldots, \overline{X_n})\|_{R_n^{[k]}} \leqslant \|p\|_{\mathcal{V}}.$$

This will follow if $R_n^{[k]} \in \mathcal{V}$.

By Corollary 4 and Proposition 1, we have algebraic isomorphisms

$$R_n^{[k]} \cong M_{n,0}^{[k]} \cong M_n/\overline{M_{n,0}^{(>k)}}.$$

Since $R_n^{[k]}$ and $M_n/\overline{M_{n,0}^{(>k)}}$ are finite-dimensional, they are isomorphic as Banach algebras; but $M \in \mathcal{V}$, so $M_n \in \mathcal{V}$, so $M_n/\overline{M_{n,0}^{(>k)}} \in \mathcal{V}$ and $\mathcal{V}$ is closed under Banach algebra isomorphisms, so $R_n^{[k]} \in \mathcal{V}$ and the result follows.

B*-algebras

A similar theorem holds for Banach *-algebras, but the polynomials are polynomials in the variables and their adjoints and the subalgebras are all *-subalgebras.

B*-algebras form a variety: what are the defining laws?

Theorem 8 (Corollary of Vidav–Palmer) *A Banach *-algebra A (with 1) is a B*-algebra iff*

$$\left.\begin{array}{rcl} \|\exp(it(x+x^*))\| & \leqslant & 1 \\ \|\exp(t(x-x^*))\| & \leqslant & 1 \end{array}\right\} \qquad (x \in A,\ t \in \mathbb{R}). \tag{3}$$

Let

$$E_n(X) = \sum_{j=0}^{n} \frac{X^j}{j!} \qquad (n = 1, 2, 3, \ldots)$$

and

$$\varepsilon_{n,t} = \sum_{j=n+1}^{\infty} \frac{|2t|^j}{j!} \qquad (n = 1, 2, 3, \ldots).$$

Then (3) is equivalent to

$$\left.\begin{array}{rcl} \|E_n(it(x+x^*))\| & \leqslant & 1 + \varepsilon_{n,t} \\ \|E_n(t(x-x^*))\| & \leqslant & 1 + \varepsilon_{n,t} \end{array}\right\} \qquad (x \in A_1) \tag{4}$$

for all $t \in \mathbb{R},\ n \in \mathbb{N}$.

Problem area 1 — Unital Banach algebras

Definition 14 A **unital Banach algebra** is a Banach algebra with an identity element e such that $\|e\| = 1$.

We can easily rewrite the definition of variety for unital Banach algebras. A **variety of unital Banach algebras** is a class of unital Banach algebras closed under products, quotients by closed ideals and unital subalgebras (meaning: subalgebras containing the identity element of the original algebra).

This definition fits with a more general theory [3], and we get a version of Birkhoff's Theorem (using unital polynomials). Unfortunately, it is not clear how a theory of semivarieties should go. The problem is that not all isomorphisms are isometries, but we do not have a plentiful supply of non-isometric isomorphisms as we do in general Banach algebra theory. The characterization of semivarieties relies heavily on the fact that one can take a Banach algebra, multiply the norm by any constant $\lambda \geqslant 1$, and obtain an isomorphic Banach algebra. The requirement $\|e\| = 1$ kills this technique for unital algebras.

Question 2 Consider the Q-algebra $C^1[0,1]$ as a unital Banach algebra. Is there an isomorphism between it and an algebra in the unital variety generated by $\mathbb{C}$?

Since $C^1[0,1]$ is a Q-algebra, there is an isomorphism between it and an IQ-algebra, and this IQ algebra has an identity, but the norm of that identity may be greater than 1.

Any IQ-algebra with an identity of norm 1 is necessarily in the unital variety generated by $\mathbb{C}$. Therefore, what we are asking is just whether $C^1[0,1]$ is isomorphic with an IQ-algebra with an identity **of norm 1**.

Given an IQ-algebra with an identity of norm 1, one can find an equivalent norm in which the identity is of norm 1, (use the multiplier norm); however, this will generally destroy the IQ property!

Problem area 2 — The finite basis problem

Definition 15 We say that a variety $\mathcal{V}$ is **finitely based** if it is defined by finitely many laws. That is, there are polynomials $p_1, \ldots, p_n$ and constants $K_1, \ldots, K_n$ such that

$$\mathcal{V} = \{A : \|p_i\|_A \leqslant K_i \qquad (1 \leqslant i \leqslant n)\}.$$

BEWARE: not the same as **finitely generated**: every variety is finitey generated.

Question 3 Are the varieties IQ, IR finitely based?

These are basic questions. To ask whether IQ is finitely based is to ask whether there is a finite set of polynomials inequalities holding in $\mathbb{C}$ from which all other polynomials inequalities holding in $\mathbb{C}$ may be deduced.

Theorem 9 (PGD) *There exists a variety which is not finitely based. (Constructed using laws of the form*

$$\|X_1 \cdots X_n\|_A \leqslant f(n),$$

for a rapidly decreasing function f.)

Remark 7 One can also ask for small, not necessarily finite, sets of laws defining a variety — e.g. polynomials in $< n$ variables or polynomials of degree $< k$.

The finite basis problem in algebra

In the algebraic situation, a variety $\mathcal{V}$ is finitely based iff every chain of varieties

$$\mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \dots$$

with

$$\bigcap \mathcal{V}_i = \mathcal{V}$$

is eventually constant. This does not hold for varieties of Banach algebras: the varieties $\mathcal{V}_n$ defined by the polynomial identities $\|xy\| \leqslant 1/n$ form a strictly decreasing sequence with intersection the variety $\mathcal{V} = \mathcal{N}_2$ defined by the polynomial identity $xy = 0$, which is finitely-based.

The finite basis problem for groups was posed by B. H. Neumann in his doctoral thesis in 1935 and solved negatively by Vaughan-Lee in 1969. A simply stated example of a variety without a finite basis is that with basis

$$x_1^8 \equiv 1, \quad (x_1^2 x_2^2)^4 \equiv 1, \dots, (x_1^2 \dots x_n^2)^4 \equiv 1, \dots.$$

This followed various positive partial results such as:

Theorem 10

(R. C. Lyndon) Every variety of nilpotent groups has a finite basis.

The finite basis problem for associative algebras over fields of characteristic zero (Specht's problem) was solved affirmatively by A. R. Kemer [9].

Summary

Modelled on the theory of varieties in pure algebra, the theory of varieties of Banach algebras throws up interesting functional-analytic problems.

To challenge our understanding of the theory of varieties of Banach algebras, we set a 'Test Question'.

We extended the theory, introducing two graded Banach algebras analogous to the 'relatively free algebras' of pure algebra: R carrying just algebraic information; M carrying topological and algebraic information. The utility of these concepts was demonstrated by using the relationship between them to answer the 'Test Question'.

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e-mail:P.Dixon@sheffield.ac.uk
WWW: http://www.shef.ac.uk/∼pm1pgd/
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Dept. of Pure Mathematics,
University of Sheffield,
SHEFFIELD, S3 7RH.