PMA/324 Chaos January 2008 — Solutions

1. (i) [8 marks: bookwork] *Define the terms* **periodic point**, **period**, **order** *(of a periodic point) and prove that the order of a periodic point divides every period of that point.*

Given a function $f : X \to X$, we say that a point $x \in X$ is a **periodic point** if there is a number $k \in \mathbb{Z}^+$ such that $f^k(x) = x$. Any such number k is called a **period** of x. The smallest such k is the **order** of x.

Let x be periodic for f with order a and let k be any period of x. Let $k = qa + r$ with $0 \leqslant r < a$. Then

$$\begin{aligned} x &= f^k(x) \\ &= f^r(f^{qa}(x)) \\ &= f^r((f^a)^q(x)) \\ &= f^r(x), \text{ since } f^a(x) = x. \end{aligned}$$

Since a is, by definition the smallest positive integer with $f^a(x) = x$, we must have $r = 0$, i.e. a divides k.

1. (ii) [5 marks: unseen problem] *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = x^3 - x \qquad (x \in \mathbb{R}).$$

Find the fixed points of f and classify them as attracting, repelling or non-hyperbolic.

To find the fixed points of $f(x) = x^3 - x$, we solve

$$x^3 - x = x,$$

and obtain

$$x = 0, \pm\sqrt{2}.$$

To classify these points, we compute $f'(x) = 3x^2 - 1$.

$$f'(x) = \begin{cases} -1 & \text{at } x = 0 : \text{ non-hyperbolic;} \\ 5 > 1 & \text{at } x = \pm\sqrt{2} : \text{ repelling.} \end{cases}$$

1. (iii) [8 marks: bookwork] *Explain, briefly, the changes that occur at the parameter values $\mu = 3$, $\mu = 1 + \sqrt{6}$, $\mu = 1 + \sqrt{8}$ and $\mu = 4$ in the dynamics of the function F_μ defined by*

$$F_\mu(x) = \mu x(1 - x) \qquad (x \in \mathbb{R}).$$

For $1 < \mu < 3$, the non-zero fixed point p_μ of F_μ is attracting. Then 3 is the point at which it becomes non-hyperbolic and then, for $\mu > 3$, repelling. At 3, an attracting 2-cycle is born.

At $\mu = 1 + \sqrt{6}$, the 2-cycle becomes repelling and an attracting 4-cycle is born.

After this, there is a period-doubling cascade, followed by a dense set of windows of periodicity of which the largest is the period 3 window, which starts at $\mu = 1 + \sqrt{8}$. This is again followed by a period doubling cascade.

For $\mu \leqslant 4$, F_μ maps $[0, 1] \to [0, 1]$. For $\mu > 4$ this is no longer the case, and there is a Cantor-like set $C \subset [0, 1]$ such that $F : C \to C$ on which the interesting dynamics occurs. Outside C, the iterates tend to $-\infty$.

1. (iv) [4 marks: unseen problem.] *For which of the following numbers N is it true that every continuous function $\mathbb{R} \to \mathbb{R}$ which has a periodic point of order 2008 has a periodic point of order N:*

$$1, \quad 2, \quad 3, \quad 4, \quad 2000, \quad 2007, \quad 2010, \quad 2024?$$

Justification of your answer is not required.

This is true for $N = 1, 2, 4, 2000, 2024$, but false for $N = 3, 2007, 2010$.

Calculation: the *Sarkovskii ordering* is: $3 \prec 5 \prec 7 \prec 9 \prec \ldots 6 \prec 10 \prec 14 \prec 18 \prec \ldots 12 \prec 20 \prec 28 \prec \ldots 24 \prec 40 \prec \ldots \ldots \quad \ldots 64 \prec 32 \prec 16 \prec 8 \prec 4 \prec 2 \prec 1$.

The number 2008 occurs in the fourth group: odd multiples of 8 greater than 8. The values of N for which the assertion is true are those following 2008 in the Sarkovskii ordering. These are

- the odd multiples of 8 greater than 8 and greater than 2008, whence $N = 2024$,
- the odd multiples of 16 greater than 16, whence $N = 2000$, and
- the powers of 2, whence $N = 1, 2, 4$.

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2. (i) [6 marks: bookwork] *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Explain what is meant by saying that a periodic point of f of order m is (a)* **attracting**, *(b)* **repelling**, *(c)* **non-hyperbolic**, *(d)* **weakly attracting**.

The definitions are the same as those for a fixed point, but with f^m in place of f throughout. Thus:

we say that a periodic point p of order m is *attracting* if $|(f^m)'(p)| < 1$, *repelling* if $|(f^m)'(p)| > 1$, and *non-hyperbolic* if $|(f^m)'(p)| = 1$.

The point p is *weakly attracting* if, for some $\delta > 0$, every $x \in B(p, \delta)$, has $(f^m)^k(x) \to p$ as $k \to \infty$.

2. (ii) [12 marks: bookwork] *Prove, from your definitions, that if $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable then every attracting periodic point of f is weakly attracting.*

We first observe that since attracting and weakly attracting periodic points for f are the same as attracting and weakly attracting fixed points for f^m, it suffices to prove this result for fixed points and then apply it to f^m, which is continuously differentiable since f is .

Let p be an attracting fixed point, and let $k = (1 + |f'(p)|)/2$ and $\varepsilon = (1 - |f'(p)|)/2$.

Since f' is continuous, $|f'|$ is continuous, so there exists $\delta > 0$ such that

$$|f'(p)| - \varepsilon < |f'(y)| < |f'(p)| + \varepsilon = k < 1$$

for all $y \in (p - \delta, p + \delta)$.

By the Mean Value Theorem, if $0 < |x - p| < \delta$, then there exists y between x and p such that

$$\frac{f(x) - f(p)}{x - p} = f'(y) < k.$$

So $|f(x) - f(p)| \leqslant k|x - p|$ for all x in $(p - \delta, p + \delta)$,
so

so $$|f(x) - p| \leqslant k|x - p| < \delta \qquad (x \in (p - \delta, p + \delta)),$$

Therefore $f(x) \in (p - \delta, p + \delta)$, so

so $$|f^2(x) - p| \leqslant k|f(x) - p| \leqslant k^2|x - p| < \delta \qquad (x \in (p - \delta, p + \delta)),$$

et cetera:

$$|f^n(x) - p| \leqslant k^n|x - p| \qquad (x \in (p - \delta, p + \delta),\ n = 1, 2, 3, \ldots).$$

So $f^n(x) \to p$, for all x in $(p - \delta, p + \delta)$; i.e. p is weakly attracting.

2. (iii) [7 marks: unseen problem] *For $\mu > 0$, show that the function $f_\mu(x) = e^{-x} + x - \mu$ has a unique fixed point and classify it under the headings 'attracting', 'repelling' and 'non-hyperbolic', making it clear for which values of μ it falls under the various headings. What happens if $\mu \leqslant 0$?*

To find the fixed points of $f_\mu(x) = e^{-x} + x - \mu$, we solve

$$e^{-x} + x - \mu = x$$

and obtain $e^{-x} = \mu$, $x = -\ln\mu$. To classify this point, we compute

$$\begin{aligned} f'_\mu(x) &= -e^{-x} + 1 \\ &= -e^{\ln\mu} + 1, \\ &= -\mu + 1. \end{aligned}$$

The fixed point is
attracting if $-1 < -\mu + 1 < +1$, i.e. if $0 < \mu < 2$; *repelling* if $-\mu + 1 < -1$, i.e. if $\mu > 2$; *non-hyperbolic* if $-\mu + 1 = -1$, i.e. if $\mu = 2$.

If $\mu \leqslant 0$, then

$$e^{-x} + x - \mu > x$$

for all x, because $e^{-x} > 0$, and so there are no fixed points.

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3. (i) [7 marks: bookwork] *Give the* **Sarkovskii ordering** *of the positive integers and state* ***Sarkovskii's Theorem.***
DEFINITION The **Sarkovskii ordering** of the positive integers is the following linear ordering.
$3 \prec 5 \prec 7 \prec 9 \prec \ldots 6 \prec 10 \prec 14 \prec 18 \prec \ldots 12 \prec 20 \prec 28 \prec \ldots 24 \prec 40 \prec \ldots\ldots \quad \ldots 64 \prec 32 \prec 16 \prec 8 \prec 4 \prec 2 \prec 1.$
THEOREM (Sarkovskii's Theorem) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a periodic point of order n, then f has periodic points of all orders greater than n in the Sarkovskii ordering. Further, for each positive integer n there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ which has a periodic point of order n, but no periodic points of order smaller than n in the Sarkovskii ordering.
3. (ii) [14 marks: example based on bookwork set as homework] *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$\begin{aligned} f(x) &= 3 \quad (x \leqslant 0), \\ f(1) &= 6, \\ f(2) &= 5, \\ f(3) &= 4, \\ f(4) &= 2, \\ f(5) &= 1, \\ f(x) &= 0 \quad (x \geqslant 6), \end{aligned}$$

with f *linear on the intervals* $(0,1), (1,2), (2,3), (3,4), (4,5), (5,6)$. *Show that* f *has periodic points of order 7 but no periodic points of order 5.*

Clearly $(0, 3, 4, 2, 5, 1, 6)$ is a 7-cycle for f. Clearly, also, there are no periodic points for f outside [0,6], since all these points are mapped into [0,6], from which they can never return to the starting point. Now

$$[2,3] \xrightarrow{f} [4,5] \xrightarrow{f} [1,2] \xrightarrow{f} [5,6] \xrightarrow{f} [0,1] \xrightarrow{f} [3,6] \xrightarrow{f} [0,4] \xrightarrow{f} [2,6] \xrightarrow{f} [0,5] \xrightarrow{f} [1,6] \xrightarrow{f} [0,6].$$

Therefore,

$$\begin{aligned} f^5([2,3]) &= [3,6], \\ f^5([4,5]) &= [0,4], \\ f^5([1,2]) &= [2,6], \\ f^5([5,6]) &= [0,5], \\ f^5([0,1]) &= [1,6], \end{aligned}$$

so there are no fixed points for f^5 outside [3,4], except possibly the points 0,1,2,3,4,5,6, but they are periodic points of f of order 7 and so cannot be fixed points for f^5. There remains the possibility that

f^5 has a fixed point in (3,4). Indeed it does: there is a fixed point of f in (3,4). However, each of the maps

$$f : [3,4] \to [2,4], \quad [2,4] \to [2,5], \quad [2,5] \to [1,5], \quad [1,5] \to [1,6], \quad [1,6] \to [0,6],$$

is monotonic decreasing. Therefore their composition

$$f^5 : [3,4] \to [0,6]$$

is monotonic decreasing and so can have no more than one fixed point. Thus f^5 has no fixed point other than the fixed point of f. Therefore f has no periodic points of order 5.

3 (iii) [4 marks: bookwork] *Define a function $F : \mathbb{R} \to \mathbb{R}$ which has periodic points of order 14 but none of order 10. You do not need to prove your assertion and you may define F in terms of the function f in (ii).*

We define F as the appropriate double of $f : [0,6] \to [0,6]$, extended to $\mathbb{R}$; i.e.

$$F(x) = \begin{cases} \frac{5}{6} & (x < 0) \\ \frac{2}{3} + \frac{1}{18} f(18x) & (0 \leqslant x \leqslant \frac{1}{3}) \\ \alpha x + \beta & (\frac{1}{3} < x < \frac{2}{3}) \\ x - \frac{2}{3} & (\frac{2}{3} \leqslant x \leqslant 1) \\ \frac{1}{3} & (x > 1), \end{cases}$$

where the constants α, β are chosen to make F continuous at 1/3 and 2/3. (In fact, $\alpha = -2$ and $\beta = \frac{4}{3}$.)

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4. (i) [3 marks: bookwork] *Explain what is meant by a* **topological conjugacy** *between two dynamical systems.*

The mapping h is a **topological conjugacy** between the two dynamical systems (f, X) and (g, Y) if h is a homeomorphism $X \to Y$ with $hf = gh$.

4. (ii) [5 marks: bookwork] *Prove from your definition that, under a topological conjugacy, the image of a periodic point is a periodic point of the same order.*

We observe that

$$hf^n = (hf)f^{n-1} = (gh)f^{n-1} = g(hf)f^{n-2} = g^2hf^{n-2} = \ldots = g^n h$$

for $n = 1, 2, 3, \ldots$. Therefore

$$f^n(p) = p \;\Rightarrow\; g^n(h(p)) = h(f^n(p)) = h(p).$$

Conversely,

$$g^n(h(p)) = h(p) \;\Rightarrow\; f^n(p) = h^{-1}(h(f^n(p))) = h^{-1}(g^n(h(p))) = h^{-1}(h(p)) = p.$$

Thus p is periodic of period n iff $h(p)$ is periodic of period n. Since the set of periods of p and $h(p)$ are the same, the least periods (i.e. orders) are the same.

4. (iii) [3+3+3 marks: unseen problems] *For each of the following pairs of dynamical systems $((f_i, \mathbb{R}), (g_i, \mathbb{R}))$, decide whether or not they are topologically conjugate and justify your answer.*
(a) $f_1(x) = x^3$, $g_1(x) = -x^3$;
(b) $f_2(x) = x^2$, $g_2(x) = -x^2$;
(c) $f_3(x) = -x$, $g_3(x) = -x^3$;

(a) The function f_1 has three fixed points on $\mathbb{R}$, namely 0, ± 1, whereas g_1 has one fixed point, namely 0. Therefore they can not be topologically conjugate dynamical systems.

(b) Let $h : X_2 \to Y_2$ be defined by $h(x) = -x$. [You could try the standard routine: let $h(x) = ax+b$ and then find a, b to give the desired result. It's easier to guess.] Then h is a homeomorphism. Moreover $h(f_2(x)) = -x^2 = -(-x)^2 = g_2(h(x))$. Thus h is a topological conjugacy between the two systems.

(c) For the function f_3, every point is periodic with period two: the point 0 is fixed and all the points other than 0 are periodic of order two. This is not true of the function g_3, since $g_3^2(2) = g_3(-8) = 512 \neq 2$. Therefore they can not be topologically conjugate dynamical systems. [In fact, g_3 has one fixed point, namely 0, and two periodic points of order two, namely ± 1.]

4. (iv) [8 marks: unseen problem] *For $\lambda > 0$, let the dynamical systems $D_\lambda = (f_\lambda, \mathbb{R}^+)$ on the set $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geqslant 0\}$ be defined by*

$$f_\lambda(x) = \lambda x \qquad (x \in \mathbb{R}^+).$$

Prove that the dynamical systems D_λ and D_μ are topologically conjugate if and only if either $\lambda, \mu \in (1, \infty)$ or $\lambda, \mu \in (0, 1)$ or $\lambda = \mu = 1$. [Hints: in one direction, consider homeomorphisms of the form $x \mapsto x^\alpha$ where $\alpha > 0$; in the other direction, you may use without proof the fact that topological conjugacies map weakly attracting periodic points to weakly attracting periodic points.]

Let $h(x) = x^\alpha$ with $\alpha > 0$. Then

$$\begin{aligned} h(f_\lambda(x)) = f_\mu(h(x) \quad &\Longleftrightarrow \quad (\lambda x)^\alpha = \mu x^\alpha \\ &\Longleftrightarrow \quad \lambda^\alpha = \mu \\ &\Longleftrightarrow \quad \alpha \ln \lambda = \ln \mu \\ &\Longleftrightarrow \quad \alpha = \frac{\ln \mu}{\ln \lambda}. \end{aligned}$$

Thus, provided that $\ln \lambda$ and $\ln \mu$ are either both positive or both negative, we have a topological conjugacy between D_λ and D_μ.

To show that D_1 is not topologically conjugate to any other D_λ we observe that all points of $\mathbb{R}^+$) are fixed for D_1, but only 0 is fixed for D_λ when $\lambda \neq 1$. To show that no D_λ for $\lambda > 1$ is topologically conjugate to a D_λ with $\lambda < 1$ we observe that the single fixed point 0 is weakly attracting for $\lambda < 1$, but not weakly attracting for $\lambda > 1$, since $f_\lambda^n(x) = \lambda^n x$ which tends to 0 or ∞ according as to whether λ is less than or greater than 1.

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5. (i) [4 marks: bookwork] *Define the* **Julia sets** *J_c and the* **Mandelbrot set** *M for the family of quadratic maps $Q_c : \mathbb{C} \to \mathbb{C}$ ($c \in \mathbb{C}$) given by $Q_c(z) = z^2 + c$ ($z \in \mathbb{C}$).*

For $c \in \mathbb{C}$, the **Julia set** J_c of the mapping Q_c is the closure of the set of all repelling periodic points of Q_c.

The **Mandelbrot set,** M is the set of all $c \in \mathbb{C}$ for which J_c is connected.

5. (ii) [4 marks: bookwork] *State a theorem giving two characterizations of M in terms of the behaviour of the sequence $(Q_c^n(0))$.*

(a) $Q_c^n(0) \to \infty$ iff $c \notin M$;

(b) $|Q_c^n(0)| \leqslant 2$ for all n iff $c \in M$.

5. (iii) [11 marks: bookwork] *Show that Q_c has an attracting fixed point if and only if c lies inside a certain cardioid.*

If p is a fixed point of Q_c, then $Q_c(p) = p$; i.e.

$$p^2 + c = p.$$

Since $Q_c'(p) = 2p$, the fixed point p is attracting if and only if $|p| < \frac{1}{2}$. Therefore the set M_0 of points c for which Q_c has an attracting fixed point is

$$M_0 = \{c : c = p - p^2, \quad |p| < \frac{1}{2}\} = g(D),$$

where $g : z \mapsto z - z^2$ and $D = B(0; 1/2)$.

The set $g(\partial D)$ is the curve

$$c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta} \qquad (0 \leqslant \theta \leqslant 2\pi),$$

which is the desired cardioid.

Now g is injective on $\overline{D}$; for if $p - p^2 = q - q^2$, then $(p-q)(1-p-q) = 0$, so either $p = q$ or $p+q = 1$. However, if $|p| \leqslant 1/2$ and $|q| \leqslant 1/2$, then $p + q = 1$ implies $p = q = 1/2$. Thus, in either case, $p = q$.

To show that every point of $g(D)$ lies inside the cardioid, we begin by noting that $g(0) = 0$ which is inside the cardioid. Suppose there is some z with $|z| < 1/2$ and $g(z)$ outside the cardioid. Then the straight line from 0 to z would be mapped by g to a curve joining 0 and $g(z)$, which would have to meet the cardioid at a point $g(z_1) = g(z_2)$ where z_1 is on the line from 0 to z and $z_2 \in \partial D$. Therefore $|z_1| < |z_2|$, so $z_1 \neq z_2$ contradicting the injectivity of g.

The converse assertion, that every point inside the cardioid is in $g(D)$, is intuitively obvious: because as r increases from 0 to 1/2 the curve $g(re^{i\theta})$ $(0 \leqslant \theta \leqslant 2\pi)$ changes continuously from the point 0 to the cardioid and 'so' sweeps out all the points in between.

(This is the 'proof' given in lectures, where it is noted that it is not a formal proof; a full proof would require more plane set topology than can be assumed in this course.)

5. (iv) [6 marks: unseen problem; bookwork; unseen problem] *Which of the following points lie in M? Justify your answers: (a) $(1+i)/8$; (b) i; (c) $1-i$.*

(a) The point $(1+i)/8$ has modulus $\sqrt{2}/8 < 1/4$ and so inside the fixed disc (radius 1/4) in the classical construction of the cardioid and hence clearly inside the region discussed in (iii). Therefore $(1+i)/8 \in M$.

(b) We have: $Q_i(0) = i$, $Q_i^2(0) = i^2 + i = -1 + i$, $Q_i^3(0) = (-1+i)^2 + i = -i$, $Q_i^4(0) = (-i)^2 + i = -1 + i = Q_i^2(0)$. Thus the sequence $(Q_c^n(0))$ is eventually periodic of order 2, and, in particular, $Q_c^n(0) \not\to \infty$. Therefore $c \in M$.

(c) For $c = 1 - i$, we have $Q_c(0) = 1 - i$ and

$$Q_c^2(0) = (1-i)^2 + 1 - i = 1 - 2i - 1 + 1 - i = 1 - 3i.$$

Therefore $|Q_c^2(0)| = \sqrt{10} > 2$. It follows from (ii)(b) that $c \notin M$.