PMA/324 Chaos January 2009 — Solutions

1. (i) [2 marks: bookwork] *Let X be a set, $f : X \to X$ a function and m a positive integer. What is meant by saying that a point $x \in X$ is* ***periodic*** *of* ***order*** *m?*
To say that $x \in X$ is periodic of order m means that $f^m(x) = x$, but there is no $k < m$ with $f^k(x) = x$.

1. (ii) [6 marks: unseen problem] *Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $f : X \to X$ be defined by*

$$f(1) = 2 \qquad f(2) = 3 \qquad f(3) = 4 \qquad f(4) = 2$$

$$f(5) = 6 \qquad f(6) = 7 \qquad f(7) = 8 \qquad f(8) = 5.$$

List the periodic and non-periodic points for f, and give the orders of the periodic points. Do the same for the periodic and non-periodic points for f^2.

$2, 3, 4$ are all periodic for f of order 3; 1 is not periodic for f and $5, 6, 7, 8$ are periodic for f of order 4.

$2, 3, 4$ are all periodic for f^2 of order 3; 1 is not periodic for f^2 and $5, 6, 7, 8$ are periodic for f^2 of order 2.

1. (iii) [7 marks: unseen problem] *Find the fixed points of the function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = 4x^3 + 3x^2$$

and classify them under the headings 'attracting', 'repelling' and 'non-hyperbolic'.

To find the fixed points of $f(x) = 4x^3 + 3x^2$, we solve

$$4x^3 + 3x^2 = x,$$

i.e.

$$4x^3 + 3x^2 - x = 0$$

$$x(4x - 1)(x + 1) = 0$$

and obtain

$$x = 0, +1/4, -1.$$

To classify these points, we compute $f'(x) = 12x^2 + 6x$.

$$f'(x) = \begin{cases} 0 & \text{at } x = 0 \text{ attracting;} \\ 9/4 & \text{at } x = 1/4 \text{ repelling;} \\ 6 & \text{at } x = -1 \text{ repelling.} \end{cases}$$

1. (iv) [4 marks: bookwork + unseen problem.] *State the Intermediate Value Theorem and use it to prove that every continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.*

Intermediate Value Theorem: if $g : [a, b] \to \mathbb{R}$ is a continuous function and $y \in [g(a), g(b)]$, then there exists $c \in [a, b]$ such that $y = g(c)$.

We apply this to the function $g(x) = f(x) - x$, with $a = 0, b = 1$. Then g is continuous and $g(0) = f(0) - 0 \geqslant 0$ and $g(1) = f(1) - 1 \leqslant 0$. Let $y = 0$; then there must exist $c \in [0, 1]$ with $f(c) - c = g(c) = y = 0$; i.e. c is a fixed point for F.

1. (v) [6 marks: bookwork + bookwork + unseen problem] *Which of the following statements about the function $F_\mu : \mathbb{R} \to \mathbb{R}$ given by $F_\mu(x) = \mu x(1 - x)$, for $\mu \geqslant 1$, is true and which false? Give brief justifications of your answers.*
(a) F_μ has no fixed point other than 0 when $\mu > 3$. (b) F_μ maps the unit interval [0,1] into itself if and only if $\mu \leqslant 4$. (c) F_μ has periodic points of all orders if $\mu = 1 + \sqrt{8}$.

(a) FALSE since $F_\mu(1 - 1/\mu) = \mu(1 - 1/\mu)(1 - (1 - 1/\mu)) = 1 - 1/\mu$ for all μ.

(b) TRUE since F_μ is increasing on $[0, \frac{1}{2}]$, decreasing on $[\frac{1}{2}, 1]$, and $F_\mu(\frac{1}{2}) = \mu/4$.

(c) TRUE since $1+\sqrt{8}$ is the smallest value of μ for which F_μ has a periodic point of order 3 ; by Sarkovskii's Theorem, since $F_\mu : \mathbb{R} \to \mathbb{R}$ is continuous, it follows that F_μ has periodic points of all orders.

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2. (i) [8 marks: bookwork] *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Explain what is meant by saying that a periodic point of f of order m is (a)* **attracting***, (b)* **repelling***, (c)* **non-hyperbolic***, (d)* **weakly attracting***, (d)* **weakly repelling**.

The definitions are the same as those for a fixed point, but with f^m in place of f throughout. Thus: we say that a periodic point p of order m is *attracting* if $|(f^m)'(p)| < 1$, *repelling* if $|(f^m)'(p)| > 1$, and *non-hyperbolic* if $|(f^m)'(p)| = 1$.

The point p is *weakly attracting* if, for some $\delta > 0$, every $x \in B(p,\delta)$, has $(f^m)^k(x) \to p$ as $k \to \infty$.

The point p is *weakly repelling* if, for some $\delta > 0$, for every $x \in B(p,\delta) \setminus \{p\}$ there exists k such that $(f^m)^k(x) \notin B(p,\delta)$.

2. (ii) [10 marks: unseen problem] *For all $c \in \mathbb{R}$, let $f_c : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$f_c(x) = x^2 + x - c.$$

Find the fixed points of f_c and classify them under the headings 'attracting', 'repelling' and 'non-hyperbolic', making it clear for which values of c they fall under the various headings.

The fixed points of f_c are where $x^2 + x - c = x$, i.e. $x = \pm\sqrt{c}$. For $c < 0$ there are no fixed points. For $c = 0$ there is just one fixed point 0. Then $f'(x) = 2x + 1$, so, for $c > 0$,

$$f'(+\sqrt{c}) = 2\sqrt{c} + 1 > 1,$$

so $+\sqrt{c}$ is repelling for all $c > 0$.

Now consider $-\sqrt{c}$ where $c > 0$. We have

$$f'(-\sqrt{c}) = 1 - 2\sqrt{c} < 1,$$

and

$$1 - 2\sqrt{c} > -1 \iff \sqrt{c} < 1.$$

Thus $-\sqrt{c}$ is attracting for $0 < c < 1$; likewise, for $c = 1$ it is non-hyperbolic and for $c > 1$ it is repelling.

For $c = 0$, $f'(0) = 1$, so 0 is non-hyperbolic.

2. (iii) [3 marks: unseen problem] *Show that, for $c > 1$, the point $p = -(1 + \sqrt{c-1})$ is periodic of order two and find the other periodic point of order two. (Note that you are not being asked to solve an equation to discover p and the other periodic point of order two.)*

We have

$$\begin{aligned} f(p) &= (-(1+\sqrt{c-1}))^2 - (1+\sqrt{c-1}) - c \\ &= (1 + c - 1 + 2\sqrt{c-1}) - 1 - \sqrt{c-1} - c \\ &= -1 + \sqrt{c-1}. \end{aligned}$$

and then

$$\begin{aligned} f(-1+\sqrt{c-1}) &= (-1+\sqrt{c-1})^2 - 1 + \sqrt{c-1} - c \\ &= (1 + c - 1 - 2\sqrt{c-1}) - 1 + \sqrt{c-1} - c \\ &= -1 - \sqrt{c-1} \\ &= p. \end{aligned}$$

Thus p is periodic of order two, the other periodic point of order two being $f(p) = -1 + \sqrt{c-1}$.

2. (iv) [4 marks: unseen problem] *For which values of c is p an attracting periodic point?*

Now

$$
\begin{aligned}
(f^2)'(p) = f'(f(p))f'(p) \\
&= f'(-1+\sqrt{c-1})f'(-1-\sqrt{c-1}) \\
&= (2(-1+\sqrt{c-1})+1)(2(-1-\sqrt{c-1})+1) \\
&= (-1+2\sqrt{c-1})(-1-2\sqrt{c-1}) \\
&= 1-4(c-1) \\
&= 5-4c
\end{aligned}
$$

So $(f^2)'(p) < +1$ whenever $c > 1$ and $(f^2)'(p) > -1$ iff $5-4c > -1$, i.e. $4c < 6$, $c < 3/2$. Thus p is attracting precisely when $1 < c < 3/2$.

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3. (i) [6 marks: bookwork] *Give the **Sarkovskii ordering** of the positive integers and state **Sarkovskii's Theorem**.*

DEFINITION The *Sarkovskii ordering* of the positive integers is the following linear ordering. $3 \prec 5 \prec 7 \prec 9 \prec \ldots 6 \prec 10 \prec 14 \prec 18 \prec \ldots 12 \prec 20 \prec 28 \prec \ldots 24 \prec 40 \prec \ldots\ldots \quad \ldots 64 \prec 32 \prec 16 \prec 8 \prec 4 \prec 2 \prec 1$.

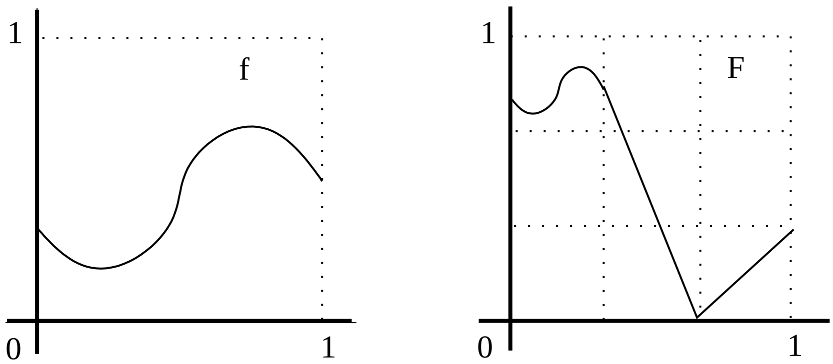
THEOREM (Sarkovskii's Theorem) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a periodic point of order n, then f has periodic points of all orders greater than n in the Sarkovskii ordering. Further, for each positive integer n there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ which has a periodic point of order n, but no periodic points of order smaller than n in the Sarkovskii ordering.

3. (ii)(a) [3 marks: bookwork set as homework] *Given a continuous function $f : [0,1] \to [0,1]$ we define the* double F *of* f *by*

$$
F(x) = \begin{cases} \frac{2}{3} + \frac{1}{3}f(3x) & (0 \leqslant x \leqslant \frac{1}{3}) \\ \alpha x + \beta & (\frac{1}{3} < x < \frac{2}{3}) \\ x - \frac{2}{3} & (\frac{2}{3} \leqslant x \leqslant 1), \end{cases}
$$

where the constants α, β are chosen to make F continuous at 1/3 and 2/3.

Draw sketch graphs of a typical f and the corresponding F to illustrate this construction.



3. (ii)(b) [5 marks: bookwork set as homework] *Show that F has a fixed point in the interval (1/3,2/3). Call it x_0. By showing that*

$$
|F(x) - x_0| \geqslant 2|x - x_0| \qquad (x \in (1/3, 2/3)),
$$

or otherwise, prove that F has no other periodic points in (1/3,2/3) except x_0.

The function F is continuous in [1/3,2/3] and

$$F(\frac{1}{3}) - \frac{1}{3} \geqslant \frac{1}{3} > 0 > -\frac{2}{3} = F(\frac{2}{3}) - \frac{2}{3}.$$

By the Intermediate Value Theorem, $F(x) - x$ has a zero in (1/3,2/3); i.e. F has a fixed point, x_0, say.

In (1/3,2/3), F has the form $F(x) = \alpha x + \beta$. We have

$$\begin{aligned} \frac{1}{3}\alpha + \beta &= F(\frac{1}{3}) \geqslant \frac{2}{3}, \\ \frac{2}{3}\alpha + \beta &= F(\frac{2}{3}) = 0. \end{aligned}$$

Hence $\alpha \leqslant -2$. Writing F in the form

$$F(x) = x_0 + \alpha(x - x_0) \qquad (\frac{1}{3} \leqslant x \leqslant \frac{2}{3}),$$

we see that,

$$|F(x) - x_0| = |\alpha|\,|x - x_0| \geqslant 2|x - x_0| \qquad (\frac{1}{3} \leqslant x \leqslant \frac{2}{3}),$$

Repeated application of this shows that, for $x \in (1/3, 2/3)$,

$$|F^n(x) - x_0| \geqslant 2^n |x - x_0|,$$

provided $F(x), \ldots, F^{n-1}(x) \in (1/3, 2/3)$.It follows that, for $x \in (1/3, 2/3) \setminus \{x_0\}$, the iterates $F^n(x)$ eventually leave (1/3,2/3). Since F maps $[0, 1/3] \cup [2/3, 1]$ into itself, the iterates can never return to (1/3,2/3) having once left it. Therefore, no such x can be periodic.

3. (ii)(c) [5 marks: bookwork set as homework] *Show that if $p \in [0, 1]$ is a periodic point of order n for f, then the points $p/3$ and $(p+2)/3$ are periodic of order $2n$ for F, and that all the periodic points of F in $[0, 1/3] \cup [2/3, 1]$ are of this form.*

If $p \in [0, 1]$, then $(p + 2)/3 \in [2/3, 1]$ and

$$F((p + 2)/3) = p/3, \qquad F(p/3) = (f(p) + 2)/3.$$

Hence

$$F^{2n}((p + 2)/3) = (f^n(p) + 2)/3, \qquad F^{2n}(p/3) = f^n(p)/3.$$

Thus, if p is period n for f, then $p/3$ and $(p + 2)/3$ are period $2n$ for F, and if either $p/3$ or $(p + 2)/3$ is period $2n$ for F, then p is period n for f. Further, we have

$$F^{2n+1}((p + 2)/3) = f^n(p)/3 \in [0, 1/3], \qquad F^{2n+1}(p/3) = (f^{n+1}(p) + 2)/3 \in [2/3, 1],$$

so no points in $[0, 1/3] \cup [2/3, 1]$ can have an odd period.

Combining these observations about possible *periods* of periodic points gives the desired result about the periodic points and their *orders*.

3. (iii) [6 marks: bookwork set as homework] *Using the 'doubling' construction, or otherwise, show that for each integer $n \geqslant 0$ there is a continuous function $f : [0, 1] \to [0, 1]$ which has periodic points of orders $1, 2, 4, \ldots, 2^n$ and only these.*

The function $f(x) = x$ has every point fixed and therefore no periodic points of higher order.

Defining $F_0 = f$ starts an inductive construction of the desired functions F_n. Given F_n with a periodic point of order 2^n but none of orders other than $1, 2, 4, 8, \ldots, 2^n$, we define F_{n+1} to be the double of F_n. Then F_{n+1} has a periodic point of order 2^{n+1}, but none of orders other than $2, 4, 8, \ldots, 2^{n+1}$ and 1.

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4. (i) [2 marks: bookwork] *Explain what is meant by a* topological conjugacy *between two dynamical systems.*

The mapping h is a *topological conjugacy* between the two dynamical systems (f, X) and (g, Y) if h is a homeomorphism $X \to Y$ with $hf = gh$.

4. (ii) [5 marks: bookwork] *Prove from your definition that, under a topological conjugacy, the image of a periodic point is a periodic point of the same order.*

We observe that

$$hf^n = (hf)f^{n-1} = (gh)f^{n-1} = g(hf)f^{n-2} = g^2hf^{n-2} = \ldots = g^nh$$

for $n = 1, 2, 3, \ldots$. Therefore

$$f^n(p) = p \;\Rightarrow\; g^n(h(p)) = h(f^n(p)) = h(p).$$

Conversely,

$$g^n(h(p)) = h(p) \;\Rightarrow\; f^n(p) = h^{-1}(h(f^n(p))) = h^{-1}(g^n(h(p))) = h^{-1}(h(p)) = p.$$

Thus p is periodic of period n iff $h(p)$ is periodic of period n. Since the set of periods of p and $h(p)$ are the same, the least periods (i.e. orders) are the same.

4. (iii)(a) [8 marks: modified bookwork; the topological conjugacy of $(F_\mu, \mathbb{R})$ and $(Q_c, \mathbb{R})$ is bookwork] *For $\mu, c \in \mathbb{C}$ the mappings $F_\mu : \mathbb{C} \to \mathbb{C}$ and $Q_c : \mathbb{C} \to \mathbb{C}$ are defined by:*

$$\begin{aligned} F_\mu(z) &= \mu z(1-z) \qquad (z \in \mathbb{C}); \\ Q_c(z) &= z^2 + c \qquad (z \in \mathbb{C}). \end{aligned}$$

Show that for every nonzero $\mu \in \mathbb{C}$ there exists $c \in \mathbb{C}$ such that the dynamical systems $(F_\mu, \mathbb{C})$ and $(Q_c, \mathbb{C})$ are topologically conjugate.

Let $H(z) = az + b$ $(z \in \mathbb{C})$, for some complex constants a, b, to be specified later, but certainly with $a \neq 0$ to ensure that H is a homeomorphism. Then

$$H(F_\mu(z)) = a\mu z(1-z) + b = -a\mu z^2 + a\mu z + b,$$

and

$$Q_c(H(z)) = (az+b)^2 + c = a^2z^2 + 2abz + b^2 + c.$$

These functions will be equal if and only if the coefficients of 1, z and z^2 match:

$$\begin{aligned} b &= b^2 + c \\ a\mu &= 2ab \\ -a\mu &= a^2. \end{aligned}$$

Since $a \neq 0$, these reduce to:

$$\begin{aligned} b &= b^2 + c \\ \mu &= 2b \\ -\mu &= a; \end{aligned}$$

i.e. given μ, we set $a = -\mu \neq 0$ and $b = \mu/2$, and we get a topological conjugacy between F_μ and Q_c, where c satisfies $b = b^2 + c$, i.e.

$$c = \frac{\mu}{2} - \left(\frac{\mu}{2}\right)^2 = \frac{\mu(2-\mu)}{4}.$$

Note: seeking a topological conjugacy in the other direction will give the map $H^{-1}(z) = -\frac{1}{\mu}z + \frac{1}{2}$.

4. (iii)(b) [2 marks: bookwork] *What are the values of c corresponding to $\mu = 1, 3$ and $1+\sqrt{6}$?*
We have

$$\begin{aligned} \mu = 1 \quad &\Rightarrow \quad c = 1/4 \\ \mu = 3 \quad &\Rightarrow \quad c = -3/4 \\ \mu = 1+\sqrt{6} \quad &\Rightarrow \quad c = (1+\sqrt{6})(1-\sqrt{6})/4 = -5/4 \end{aligned}$$

4. (iv) [2+3+3 marks: 'unseen' problems, but the same problems for $\mathbb{R}$ instead of $\mathbb{C}$ were set in January 2008] *For each of the following pairs of dynamical systems $((f_i, \mathbb{C}), (g_i, \mathbb{C}))$, decide whether or not they are topologically conjugate and justify your answer.*
(a) $f_1(z) = z^2$, $g_1(z) = -z^2$;
(b) $f_2(z) = -z$, $g_2(z) = -z^3$;
(c) $f_3(z) = z^3$, $g_3(z) = -z^3$.

(a) Let $h : \mathbb{C} \to \mathbb{C}$ be defined by $h(z) = -z$. [You could try the standard routine: let $h(z) = az + b$ and then find a, b to give the desired result. It's easier to guess.] Then h is a homeomorphism. Moreover $h(f_1(z)) = -z^2 = -(-z)^2 = g_1(h(z))$. Thus h is a topological conjugacy between the two systems.

(b) For the function f_2, every point is periodic with period two: the point 0 is fixed and all the points other than 0 are periodic of order two. This is not true of the function g_2, since $g_2^2(2) = g_2(-8) = 512 \neq 2$. Therefore they can not be topologically conjugate dynamical systems.

(c) Let $h : \mathbb{C} \to \mathbb{C}$ be defined by $h(z) = iz$. [Again, you could try the standard routine: let $h(z) = az + b$ and then find a, b to give the desired result. It's perhaps easier to guess.] Then h is a homeomorphism. Moreover $h(f_3(z)) = iz^3 = -i^3z^3 = -(iz)^3 = g_3(h(z))$. Thus h is a topological conjugacy between the two systems.

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5. (i) [4 marks: bookwork] *Define the* Mandelbrot set M *and the* Julia sets J_c *for the family of quadratic maps $Q_c : \mathbb{C} \to \mathbb{C}$ ($c \in \mathbb{C}$) given by $Q_c(z) = z^2 + c$.*

For $c \in \mathbb{C}$, the *Julia set* J_c of the mapping Q_c is the closure of the set of all repelling periodic points of Q_c.

The *Mandelbrot set,* M is the set of all $c \in \mathbb{C}$ for which J_c is connected.

5. (ii) [6 marks: bookwork] *State results which characterize M and J_c in ways which would allow them to be calculated (approximately) for display on a computer screen.*

Theorem *The set J_c is the closure of the set of all iterated inverse images of either one of the repelling fixed points of Q_c.*

[The fixed points of Q_c are determined by solving the quadratic equation $z = z^2 + c$. A fixed point p is repelling iff $|2p| = |Q_c'(p)| > 1$. Inverse images are computed by $z \mapsto \pm\sqrt{z - c}$. Plotting the iterated inverse images gives, for practical purposes, the plot of the J_c.]

Theorem

1. $Q_c^n(0) \to \infty$ iff $c \notin M$;
2. $|Q_c^n(0)| \leqslant 2$ for all n iff $c \in M$.

[To plot M, we take each value of c in turn, one corresponding to each pixel on the screen, and compute the sequence $Q_c^n(0)$ ($1 \leqslant n \leqslant N$) for some suitably large N. If, for some n, we find $|Q_c^n(0)|^2 > 4$ then $c \notin M$. Otherwise, we say $c \in M$ (this is precise only for $N = \infty$!).]

5. (iii) [3 marks: unseen problem] *Using the characterization of M you have stated determine whether or not $-1 + i \in M$;*

For $c = -1 + i$ we have

$$\begin{aligned} Q_c(0) &= c = -1 + i \\ Q_c^2(0) &= (-1+i)^2 + c = -2i - 1 + i = -1 - i \\ Q_c^3(0) &= (-1-i)^2 + c = +2i - 1 + i = -1 + 3i. \end{aligned}$$

Thus $|Q_c^3(0)| = \sqrt{10} > 2$ and so $-1+i \notin M$.

5. (iv)(a) [7 marks: bookwork] *For $c = (2+i)/8$, find the fixed points of Q_c and classify them as attracting, repelling or non-hyperbolic.*

For the fixed points, we solve $Q_c(z) = z$, i.e.

$$z^2 + \frac{2+i}{8} = z.$$

This gives

$$\begin{aligned} z &= \frac{1 \pm \sqrt{1-(2+i)/2}}{2} \\ &= \frac{1 \pm \sqrt{-i/2}}{2} \\ &= \frac{1 \pm (1-i)/2}{2} \\ &= \frac{3-i}{4}, \frac{1+i}{4}. \end{aligned}$$

Now

$$|Q_c'(z)| = |2z| = \begin{cases} \sqrt{10}/2 & \text{when } z = (3-i)/4 \\ \sqrt{2}/2 & \text{when } z = (1+i)/4. \end{cases}$$

Since $\sqrt{10}/2 > 1$ and $\sqrt{2}/2 < 1$, the fixed point $(3-i)/4$ is repelling and the fixed point $(1+i)/4$ is attracting.

5. (iv)(b) [2 mark: unseen problem] *Hence, or otherwise determine whether or not $c \in M$;*

Since Q_c has an attracting fixed point, $c = (2+i)/8$ lies in M, by a corollary of Fatou's Critical Points Theorem; in fact c is inside the cardioidal part of M.

5. (iv)(c) [3 marks: unseen problem] *Find two distinct points of J_c.*

The repelling fixed point $z = (3-i)/4$ is one point of J_c, and since $Q_c(-z) = Q_c(z) = z$, the point $-z = (-3+i)/4$ is an inverse image of it and is therefore another point of J_c.