PMA/324 Chaos January 2010 — Solutions

1. (i) [2 marks: bookwork] *Let X be a set, $f : X \to X$ a function and m a positive integer. What is meant by saying that a point $x \in X$ is* ***periodic*** *of* ***order*** *m?*
To say that $x \in X$ is periodic of order m means that $f^m(x) = x$, but there is no $k < m$ with $f^k(x) = x$.

1. (ii) [5 marks: unseen problem] *Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$\begin{aligned} f(x,y) &= (y-1, x+1) \\ g(x,y) &= (y+1, x-1). \end{aligned}$$

Show that both f and g have periodic points, but the composition gf does not.

We have

$$\begin{aligned} f^2(x,y) &= f(y-1, x+1) = (x,y) \\ g^2(x,y) &= g(y+1, x-1) = (x,y), \end{aligned}$$

so every point is periodic of period 2, [**Alternatively,** observe that the points $(\lambda, \lambda+1)$ are fixed for f and the points $(\lambda+1, \lambda)$ are fixed for g. All the other points are periodic of order 2.] However

$$gf(x,y) = g(y-1, x+1) = (x+2, y-2),$$

so

$$(gf)^n(x,y) = (x+2n, y-2n) \neq (x,y) \qquad (n = 1, 2, 3, \ldots).$$

so no points are periodic for gf.

1. (iii) [6 marks: unseen problem.] *Find the fixed points of the function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = 3x^3 - 2x^2$$

and classify them under the headings 'attracting', 'repelling' and 'non-hyperbolic'.

To find the fixed points of $f(x) = 3x^3 - 2x^2$, we solve

$$3x^3 - 2x^2 = x,$$

i.e.

$$3x^3 - 2x^2 - x = 0$$

$$x(x-1)(3x+1) = 0$$

and obtain

$$x = 0, +1, -1/3.$$

To classify these points, we compute $f'(x) = 9x^2 - 4x$.

$$f'(x) = \begin{cases} 0 & \text{at } x = 0 \text{ attracting;} \\ 7/3 & \text{at } x = -1/3 \text{ repelling;} \\ 5 & \text{at } x = +1 \text{ repelling.} \end{cases}$$

1. (iv) [6 marks: unseen problem] *For which integers $n > 1$ is it true that whenever a continuous function $f : \mathbb{R} \to \mathbb{R}$ has a periodic point of order n it necessarily has a periodic point of order $n-1$? Justify your answer.*

If n is odd, then $n-1$ is in one of the later groups in the Sarkovskii ordering, after all the odd integers greater than 1, and so f will necessarily have periodic points of order $n-1$.

If n is even and $n \geqslant 4$ then $n-1$ is in the first group in the Sarkovskii ordering, and so f will not necessarily have periodic points of order $n-1$.

If $n = 2$ then f will necessarily have a periodic point of order $n-1$ since 1 follows 2 in the Sarkovskii ordering.

1. (v) [6 marks: bookwork] *Defining $Q_c : \mathbb{C} \to \mathbb{C}$ for every $c \in \mathbb{C}$ by $Q_c(z) = z^2 + c$ ($z \in \mathbb{C}$), describe the dynamics of the function Q_0. How do the dynamics of Q_c differ from those of Q_0 when c is small but non-zero?*

The function Q_0 has an attracting fixed point at 0 with basin of attraction $\{z : |z| < 1\}$.

It also has a repelling fixed point at 1 and repelling periodic points densely distributed around the unit circle $\{z : |z| = 1\}$. The unit circle is invariant under Q_c.

For $|z| > 1$, we have $|Q_c^n(z)| \to \infty$ as $n \to \infty$.

A similar picture applies when c is small but non-zero, except that the unit circle is replaced by a quasicircle.

---

2. (i) [7 marks: bookwork] *Prove that the order of a periodic point divides every period of that point.*

Let x be periodic for f with order a and let k be any period of x. Let $k = qa + r$ with $0 \leqslant r < a$. Then

$$\begin{aligned} x &= f^k(x) \\ &= f^r(f^{qa}(x)) \\ &= f^r((f^a)^q(x)) \\ &= f^r(x), \text{ since } f^a(x) = x. \end{aligned}$$

Since a is, by definition the smallest positive integer with $f^a(x) = x$, we must have $r = 0$, i.e. a divides k.

2. (ii) [8 marks: unseen problem] *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(x, y, z) = (y, z, -x)$. Show that every point of $\mathbb{R}^3$ is periodic of period 6 and classify the periodic points according to their order.*

[**Comment:** the distinction between **period** and **order** is essential in this part and the next.]

Successive applications of f map

$$(x, y, z) \mapsto (y, z, -x) \mapsto (z, -x, -y) \mapsto (-x, -y, -z) \mapsto (-y, -z, x) \mapsto (-z, x, y) \mapsto (x, y, z),$$

so $f^6(x, y, z) = (x, y, z)$ for every $(x, y, z) \in \mathbb{R}^3$. Thus every point of $\mathbb{R}^3$ is periodic of period 6, and so order 1, 2, 3 or 6. The points of period 3 are those with $(x, y, z) = (-x, -y, -z)$, i.e. $x = y = z = 0$. Therefore (0,0,0) is the only fixed point and there are no points of order 3. The point (x, y, z) has order 2, iff $(x, y, z) = (z, -x, -y)$, i.e. $x = z = -y$. Thus the points of order 2 are the points $(\lambda, -\lambda, \lambda)$ for $\lambda \in \mathbb{R} \setminus \{0\}$.All the remaining points have order 6.

2. (iii) [10 marks: bookwork modified by replacing 2 by 3, then unseen problem.] *Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle in the complex plane and let g be the angle-**trebling** map, $g : z \mapsto z^3$. Find a general formula for the periodic points of g of **period** n, and hence identify the fixed points and the periodic points of **orders** 2 and 4.*

We have $g^n(z) = z^{3^n}$ so $z = \exp i\theta$ is periodic period n iff $\exp(3^n i\theta) = \exp i\theta$, i.e. iff $3^n\theta = \theta + 2k\pi$, for some $k \in \mathbb{Z}$; i.e. iff

$$\theta = 2k\pi/(3^n - 1)$$

for some $0 \leqslant k < 3^n - 1$. Thus the periodic points are the $(3^n - 1)$th roots of unity.

The fixed points are the periodic points of order 1, namely the square roots of unity: +1 and -1.

The periodic points of period 2 are the eighth roots of unity, so the periodic points of order 2 are all of these except ± 1, i.e.

$$\exp\left(\frac{k\pi i}{4}\right) \qquad (k = 1, 2, 3, 5, 6, 7).$$

The periodic points of period 4 are the 80th roots of unity, so the periodic points of order 4 are all of these except the eighth roots of unity, i.e.

$$\exp\left(\frac{k\pi i}{40}\right) \qquad (k = 1, 2, 3, 4, 5, 6, 7, 8, 9, \quad 11, 12, 13, 14, 15, 16, 17, 18, 19, \quad 21, \ldots, 79).$$

---

3. (i) [5 marks: bookwork]*Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Explain what is meant by saying that a fixed point of f of order m is (a)* **attracting***, (b)* **repelling***, (c)* **non-hyperbolic***, (d)* **weakly repelling**.

We say that a fixed point p is *attracting* if $|f'(p)| < 1$, *repelling* if $|f'(p)| > 1$, and *non-hyperbolic* if $|f'(p)| = 1$.

The point p is *weakly repelling* if, for some $\delta > 0$, for every $x \in B(p, \delta) \setminus \{p\}$ there exists k such that $f^k(x) \notin B(p, \delta)$.

3. (ii) [11 marks: bookwork done as homework]*Prove that every repelling fixed point of f is weakly repelling.*

Suppose that p is a fixed point of f and $|f'(p)| > 1$.

Let $k = (1 + |f'(p)|)/2$ and $\varepsilon = (|f'(p)| - 1)/2$.

[**Comment:** ε must be positive, so $\varepsilon = (1 - |f'(p)|)/2$ must be wrong.]

Since f' is continuous, $|f'|$ is continuous, so there exists $\delta > 0$ such that

$$|f'(p)| + \varepsilon > |f'(y)| > |f'(p)| - \varepsilon = k > 1$$

for all $y \in (p - \delta, p + \delta)$.

By the Mean Value Theorem, if $0 < |x - p| < \delta$, then there exists y between x and p such that

$$\left| \frac{f(x) - f(p)}{x - p} \right| = |f'(y)| > k.$$

So $|f(x) - f(p)| \geqslant k|x - p|$ for all x in $(p - \delta, p + \delta)$,
So, if $x \in (p - \delta, p + \delta)$ *and* $|f(x) - p| < \delta$, then

$$|f^2(x) - p| \geqslant k|f(x) - p| \geqslant k^2|x - p|;$$

et cetera.

[**Comment:** Note the difference between the argument here and the proof of of the corresponding theorem on attracting fixed points where $|f(x) - p| < \delta$ follows from $x \in (p - \delta, p + \delta)$.]

Thus we have

$$|f^n(x) - p| \geqslant k^n|x - p|,$$

provided that

$$x, f(x), f^2(x), \ldots, f^{n-1}(x) \in (p - \delta, p + \delta).$$

Since $k > 1$, if $x \neq p$ then $k^n|x - p| \to \infty$ as $n \to \infty$. Consequently, there must exist n such that $f^n(x) \notin U$. This is the desired result.

3. (iii) [9 marks: unseen problem]*For all $\mu > 0$, let $g_\mu : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$g_\mu(x) = \mu x e^{-x} \qquad (x \in \mathbb{R}).$$

Find the fixed points of g_μ and classify them under the headings 'attracting', 'repelling' and 'non-hyperbolic', making it clear for which values of μ they fall under the various headings.

[**Comment:** this was quite a tricky question, but generally well done.]

The fixed points of g_μ are where $\mu x e^{-x} = x$, i.e. $x = 0$ or $e^{-x} = 1/\mu$, i.e. $x = \ln \mu$.

We have

$$g'_\mu(x) = \mu(1 - x)e^{-x}.$$

Therefore

$$g'_\mu(0) = \mu,$$

so 0 is attracting for $0 < \mu < 1$, non-hyperbolic for $\mu = 1$ and repelling for $\mu > 1$.

Then

$$g'_\mu(\ln \mu) = \mu(1 - \ln \mu)e^{-\ln \mu} = 1 - \ln \mu.$$

If $\mu = 1$ then $\ln \mu = 0$ and is non-hyperbolic, as has already been shown.
If $0 < \mu < 1$, then $g'_\mu(\ln \mu) > 1$, so $\ln \mu$ is repelling.
If $1 < \mu < e^2$, then $1 > g'_\mu(\ln \mu) > -1$, so $\ln \mu$ is attracting.
If $\mu = e^2$, then $g'_\mu(\ln \mu) = -1$, so $\ln \mu$ is non-hyperbolic.
If $\mu > e^2$, then $g'_\mu(\ln \mu) < -1$, so $\ln \mu$ is repelling.

---

4. (i) [6 marks: bookwork] *Give the **Sarkovskii ordering** of the positive integers and state **Sarkovskii's Theorem**.*

DEFINITION The *Sarkovskii ordering* of the positive integers is the following linear ordering. $3 \prec 5 \prec 7 \prec 9 \prec \ldots 6 \prec 10 \prec 14 \prec 18 \prec \ldots 12 \prec 20 \prec 28 \prec \ldots 24 \prec 40 \prec \ldots\ldots \quad \ldots 64 \prec 32 \prec 16 \prec 8 \prec 4 \prec 2 \prec 1$.

THEOREM (Sarkovskii's Theorem) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a periodic point of order n, then f has periodic points of all orders greater than n in the Sarkovskii ordering. Further, for each positive integer n there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ which has a periodic point of order n, but no periodic points of order smaller than n in the Sarkovskii ordering.

[**Comment:** marks were awarded for a precise statement of the theorem, including the conditions 'f continuous' and '$f : \mathbb{R} \to \mathbb{R}$' without which the result would not be true.]

4. (ii) [12 marks: bookwork] *Let I, J be compact intervals. We write $I \xrightarrow{f} J$ if $f(I) \supseteq J$. Prove, without using Sarkovskii's Theorem, that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which has a periodic point of order 3, then f has periodic points of all orders. You may use without proof the fact that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and if $I_0, I_1, \ldots, I_k$ is a finite sequence of compact intervals with*

$$I_0 \xrightarrow{f} I_1 \xrightarrow{f} \ldots \xrightarrow{f} I_k = I_0,$$

then f^k has a fixed point p in I_0 with $f^j(p) \in I_j$ $(1 \leqslant j \leqslant k)$.

Let a be a periodic point of f of order 3. Let $b = f(a), c = f^2(a)$. Replacing a by b or c if necessary, we may assume that either $a < b < c$ or $a > b > c$. The two cases are similar: we consider the former.

Let k be a positive integer. We show that f has a periodic point of order k. For $k > 1$, let

$$\begin{aligned} I_j &= [b, c] \quad (0 \leqslant j \leqslant k-2), \\ I_{k-1} &= [a, b], \\ I_k &= [b, c]. \end{aligned}$$

For $k = 1$, let $I_0 = I_1 = [b, c]$.

Since $f(b) = c$ and $f(c) = a$ and f is continuous, $[b, c] \xrightarrow{f} [a, c]$ and so $[b, c] \xrightarrow{f} [a, b]$ and $[b, c] \xrightarrow{f} [b, c]$. Likewise, $[a, b] \xrightarrow{f} [b, c]$. Thus, for all $k \geqslant 1$,

$$I_0 \xrightarrow{f} I_1 \xrightarrow{f} \ldots \xrightarrow{f} I_k = I_0.$$

The result we are allowed to assume applies to produce a point $p \in I_0$ which is a fixed point of f^k, and therefore a periodic point of f of *period* k, with $f^j(p) \in I_j$ $(1 \leqslant j \leqslant k)$. For $k = 1$, this is enough. For $k > 1$, we must show that k is the *order* of p. Suppose the order of p is $\ell < k$. Then $k - 1 \equiv \ell - 1 \pmod{\ell}$, so $f^{\ell-1}(p) = f^{k-1}(p)$. But $f^{k-1}(p) \in I_{k-1} = [a, b]$, whilst $f^{\ell-1}(p) \in I_{\ell-1} = [b, c]$. Therefore, $f^{k-1}(p) = b$. Hence $p = f^k(p) = f(b) = c$, which is impossible if $k = 2$ because $f^2(c) \neq c$. Hence, $k > 2$, so $a = f(p) \in I_1 = [b, c]$, giving another contradiction. Therefore, k is the order of p, and the proof is complete.

4. (iii) [4 marks: bookwork] *Explain what is meant by saying that two dynamical systems (f, X) and (g, Y) are* topologically conjugate. *What can you say about about the orders of periodic points in topologically conjugate systems? (Proofs are not required.)*

Two dynamical systems (f, X) and (g, Y) are *topologically conjugate* if there is a *topological conjugacy* between them, i.e. a homeomorphism $h : X \to Y$ with $hf = gh$.

If (f, X) and (g, Y) are topologically conjugate, then for each positive integer n they have the same number of periodic points of order n. In fact, any topological conjugacy between them is an order-preserving bijection between the periodic points.

4. (iv) [3 marks: unseen problem] *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that the dynamical systems $(f, \mathbb{R})$ and $(f^2, \mathbb{R})$ are topologically conjugate. Show that if f has a periodic point of order 24 then it has periodic points of all orders.*

If f has a periodic point p of order $2n$, then $(f^2)^k(p) = p$ for $k = n$, but for no smaller k; i.e. that point is periodic for f^2 of order n. Since $(f, \mathbb{R})$ and $(f^2, \mathbb{R})$ are topologically conjugate, it follows that f has a periodic point of order n. Applying this thrice, we see that if f has a periodic point of order 24 then f has a periodic point of order 3 and hence, by (ii), periodic points of all orders.

---

5. (i) [4 marks: bookwork] *Define the* Mandelbrot set M *and the* Julia sets J_c *for the family of quadratic maps $Q_c : \mathbb{C} \to \mathbb{C}$ ($c \in \mathbb{C}$) given by $Q_c(z) = z^2 + c$.*

For $c \in \mathbb{C}$, the *Julia set* J_c of the mapping Q_c is the closure of the set of all repelling periodic points of Q_c.

The *Mandelbrot set,* M is the set of all $c \in \mathbb{C}$ for which J_c is connected.

5. (ii) [6 marks: bookwork] *Prove that the set of values of c such that Q_c has an attracting fixed point is the continuous image of a disc under a mapping which maps the perimeter of the disc onto a cardioid. (You are **not** asked to prove that the interior of the disc is mapped to the region inside the cardioid.)*

If p is a fixed point of Q_c, then $Q_c(p) = p$; i.e.

$$p^2 + c = p.$$

Since $Q_c'(p) = 2p$, the fixed point p is attracting if and only if $|p| < \frac{1}{2}$. Therefore the set M_0 of points c for which Q_c has an attracting fixed point is

$$M_0 = \{c : c = p - p^2, \quad |p| < \frac{1}{2}\} = g(D),$$

where $g : z \mapsto z - z^2$ and $D = B(0; 1/2)$.

The set $g(\partial D)$ (the image of the boundary of D) is the curve

$$c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta} \qquad (0 \leqslant \theta \leqslant 2\pi),$$

which is the equation of a cardioid traced out by a point on the circumference of a disc of radius 1/4 rolling around a fixed disc centre 0, radius 1/4.

5. (iii)(a) [7 marks: bookwork] *Draw a sketch of the Mandelbrot set M indicating four key (non-zero) points on the real axis (with their numerical values).*

The diagram is on a separate sheet. The points $c = 1/4, -3/4, -5/4, -2$ are shown.

5. (iii)(b) [3 marks: bookwork] *Explain the significance of the two largest components of the interior of M.*

The largest component is the cardioidal region discussed in the last section; for c in this region, Q_c has an attracting fixed point.

The second largest component is a circular region; for c in this region, Q_c has an attracting 2-cycle.

5. (iv) [5 marks: unseen problems] *Show that for all $c \in \mathbb{C}$, the dynamical systems $(Q_c, \mathbb{C})$ and $(Q_{\overline{c}}, \mathbb{C})$ are topologically conjugate. What consequence of this do you see in your drawing of M? (You are not asked to prove the consequence.)*

We guess that the homeomorphism $h(z) = \overline{z}$ might do the trick; and it does:

$$h(Q_c(z)) = \overline{z^2 + c} = (\overline{z})^2 + \overline{c} = (h(z))^2 + \overline{c} = Q_{\overline{c}}(h(z)).$$

The consequence for M is that $M = h(M)$, i.e. that M is symmetrical about the real axis.

PMA324 January 2010

Question 5 (iii)(a)

This is a rough sketch of the Mandelbrot set, as required, showing the main regions corresponding to an attracting fixed point, attracting 2-cycle, attracting 3-cycle. The four key points on the real axis are marked.