PMA324 Chaos 2009–10
Question Sheet 1

To be handed in on Thursday 8 October.

1. Sketch the graphs of the following functions of x: $2\sin(2\sin x)$, $\cos(\cos x)$, $1-(1-x^2)^2$, $1-(1-(1-x^2)^2)^2$. (Only *sketch* graphs are required. Use the fact that these are iterative powers of easy functions to think out, directly, how they vary as x runs from $-\infty$ to $+\infty$. Please do not use graphics calculators/MAPLE.)

2. Let X be the four-point set $\{0,1,2,3\}$ and define $f: X \to X$ by $f(0)=1$, $f(1)=2$, $f(2)=3$, $f(3)=0$. Show that 0 is periodic for each of the functions f, f^2, f^3 and f^4, and find its order (i.e. least period) in each case.

3. As in the lectures, let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and let $f : S^1 \to S^1$ be defined by $f(z) = z^2$ $(z \in S^1)$. Using the characterization of periodic points of **period** n given in the lectures, list the periodic points of **order** n for $n = 1, 2, 3, 4$. How many periodic points of orders 6 and 8 are there?

   For each n, the set of all periodic points of order n is a disjoint union of orbits. List the orbits for the periodic points you have found above for $n = 1, 2, 3, 4$.

4. Let $f : X \to X$ be any mapping on a set X. Suppose that $p \in X$ is a periodic point of order a for f and that n is a positive integer. Show that p is periodic for f^n of period r if and only if nr is a common multiple of a and n. Deduce that p is periodic for f^n with **order** $a/(a,n)$ where (a,n) denotes the highest common factor of a and n. (Remember that the lowest common multiple of a and n is $an/(a,n)$.)

PMA324 Chaos 2009–10
Question Sheet 1: Solutions

1. *Sketch the graphs of the following functions of x:* $2\sin(2\sin x)$, $\cos(\cos x)$, $1-(1-x^2)^2$, $1-(1-(1-x^2)^2)^2$.

   This will be discussed in lectures using Maple.

2. *Let X be the four-point set $\{0,1,2,3\}$ and define $f : X \to X$ by $f(0) = 1, f(1) = 2, f(2) = 3, f(3) = 0$. Show that 0 is periodic for each of the functions f, f^2, f^3 and f^4, and find its order (i.e. least period) in each case.* We have $f(0) = 1$, $f^2(0) = 2$, $f^3(0) = 0$, $f^4(0) = 0$, so 0 is periodic of order 4 for f and is a fixed point (periodic point, order 1) for f^4. Moreover, $f^2(0) = 2, f^4(0) = 0$, so 0 is periodic of order 2 for f^2. For f^3, we have $f^3(0) = 3, f^6(0) = 2, f^9(0) = 1$, $f^{12}(0) = 0$, so 0 is periodic of order 4 for f^3.

3. *As in the lectures, let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and let $f : S^1 \to S^1$ be defined by $f(z) = z^2$ $(z \in S^1)$. Using the characterization of periodic points of* **period** *n given in the lectures, list the periodic points of* **order** *n for $n = 1,2,3,4$. How many periodic points of orders 6 and 8 are there?*

   For each n, the set of all periodic points of order n is a disjoint union of orbits. List the orbits for the periodic points you have found above for $n = 1,2,3,4$.

   The periodic points of **period** n are the (2^n-1)th roots of unity:

$$\exp\left(\frac{2k\pi i}{2^n-1}\right) \qquad (k = 0,1,2,\ldots,2^n-2).$$

$n = 1$ There is just one fixed point, namely $z = 1$: its orbit is just a single point $\{1\}$.

$n = 2$ There are $2^2 - 1 = 3$ periodic points of period 2, namely

$$z = 1, \exp\left(\frac{2\pi i}{3}\right), \exp\left(\frac{4\pi i}{3}\right).$$

Of these, only the point 1 is fixed; the remaining points,

$$z = \exp\left(\frac{2\pi i}{3}\right), \exp\left(\frac{4\pi i}{3}\right),$$

are of order 2. These points are in orbits of size 2, so we have a single orbit

$$\left\{\exp\left(\frac{2\pi i}{3}\right), \exp\left(\frac{4\pi i}{3}\right)\right\}.$$

$n = 3$ There are $2^3 - 1 = 7$ periodic points of period 3, namely

$$z = 1, \exp\left(\frac{2\pi i}{7}\right), \exp\left(\frac{4\pi i}{7}\right), \exp\left(\frac{6\pi i}{7}\right), \exp\left(\frac{8\pi i}{7}\right), \exp\left(\frac{10\pi i}{7}\right), \exp\left(\frac{12\pi i}{7}\right).$$

Of these, only the point 1 is fixed; the remaining points,

$$z = \exp\left(\frac{2\pi i}{7}\right), \exp\left(\frac{4\pi i}{7}\right), \exp\left(\frac{6\pi i}{7}\right), \exp\left(\frac{8\pi i}{7}\right) \exp\left(\frac{10\pi i}{7}\right), \exp\left(\frac{12\pi i}{7}\right).$$

are of order 3. These points are in orbits of size 3. We trace the action of f on these points and find that,

$$\exp\left(\frac{2\pi i}{7}\right) \mapsto \exp\left(\frac{4\pi i}{7}\right) \mapsto \exp\left(\frac{8\pi i}{7}\right) \mapsto \exp\left(\frac{2\pi i}{7}\right),$$

so we have an orbit

$$\left\{\exp\left(\frac{2\pi i}{7}\right), \exp\left(\frac{4\pi i}{7}\right), \exp\left(\frac{8\pi i}{7}\right)\right\}.$$

Likewise (or just because it's all that's left), the other orbit is

$$\left\{\exp\left(\frac{6\pi i}{7}\right), \exp\left(\frac{12\pi i}{7}\right), \exp\left(\frac{10\pi i}{7}\right)\right\}.$$

$n = 4$ There are $2^4 - 1 = 15$ periodic points of period 4, namely

$$z = 1, \exp\left(\frac{2\pi i}{15}\right), \exp\left(\frac{4\pi i}{15}\right), \exp\left(\frac{2\pi i}{5}\right), \exp\left(\frac{8\pi i}{15}\right),$$

$$\exp\left(\frac{2\pi i}{3}\right), \exp\left(\frac{4\pi i}{5}\right), \exp\left(\frac{14\pi i}{15}\right), \exp\left(\frac{16\pi i}{15}\right), \exp\left(\frac{6\pi i}{5}\right),$$

$$\exp\left(\frac{4\pi i}{3}\right) \exp\left(\frac{22\pi i}{15}\right), \exp\left(\frac{8\pi i}{5}\right), \exp\left(\frac{26\pi i}{15}\right), \exp\left(\frac{28\pi i}{15}\right).$$

Of these, only the point 1 is fixed, and the points

$$z = \exp\left(\frac{2\pi i}{3}\right), \exp\left(\frac{4\pi i}{3}\right),$$

are of order 2. The remaining 12 points fit into 3 disjoint orbits, each or size 4, as follows

$$\left\{\exp\left(\frac{2\pi i}{15}\right), \exp\left(\frac{4\pi i}{15}\right), \exp\left(\frac{8\pi i}{15}\right), \exp\left(\frac{16\pi i}{15}\right)\right\},$$

$$\left\{\exp\left(\frac{6\pi i}{15}\right), \exp\left(\frac{12\pi i}{15}\right), \exp\left(\frac{24\pi i}{15}\right), \exp\left(\frac{6\pi i}{5}\right)\right\},$$

$$\left\{\exp\left(\frac{14\pi i}{15}\right), \exp\left(\frac{28\pi i}{15}\right), \exp\left(\frac{26\pi i}{15}\right), \exp\left(\frac{22\pi i}{5}\right)\right\}.$$

$n = 6$ There are $2^6 - 1 = 63$ periodic points of period 6. Of these, 1 is fixed, 2 are of order 2, and 6 are of order 3. That leaves 54 of order 6, which split into 9 disjoint orbits.

$n = 8$ There are $2^8 - 1 = 255$ periodic points of period 8. Of these, 1 is fixed, 2 are of order 2, and 12 are of order 4. That leaves 240 of order 8, which split into 30 disjoint orbits.

4. *Let $f : X \to X$ be any mapping on a set X. Suppose that $p \in X$ is a periodic point of order a for f and that n is a positive integer. Show that p is periodic for f^n of period r if and only if nr is a common multiple of a and n. Deduce that p is periodic for f^n with* **order** *$a/(a, n)$ where (a, n) denotes the highest common factor of a and n. (Remember that the lowest common multiple of a and n is $an/(a, n)$.)*

We have

$$\begin{aligned} p \text{ periodic for } f^n \text{ period } r &\iff (f^n)^r(p) = p \\ &\iff f^{nr}(p) = p \\ &\iff p \text{ periodic for } f \text{ period } nr \\ &\iff nr \text{ is a multiple of } a. \end{aligned}$$

Since nr is already a multiple of n, the last line is equivalent to nr being a common multiple of a and n.

Thus p is a periodic point for f^n. Its periods are the r such that nr is a common multiple of a and n. Its order is the least such r. Now all the common multiples of a and n are of the form nr for some r, so the order of p for f^n is the r such that nr is the least common multiple of a and n. Thus

$$\text{order} = \frac{an/(a, n)}{n} = \frac{a}{(a, n)}.$$

PMA324 Chaos 2009–10
Question Sheet 2

To be handed in on Thursday 15 October.

1. Let $f : X \to X$ be a continuous function on a set $X \subseteq \mathbb{R}$.

   (a) Show that the set $\mathrm{Fix}(f)$ of all fixed points of f is closed. (Recall that a set $S \subseteq \mathbb{R}^n$ is closed if and only if every sequence in S that converges to a point in $\mathbb{R}^n$ has its limit in S.)

   (b) Hence, or otherwise, show that, for each positive integer n, the set $\mathrm{Per}_n(f)$ of all periodic points of period n is closed. (You may use without proof the fact that the composition $f \circ g$ of two continuous functions is necessarily continuous.)

   (c) By considering the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -x$, or otherwise, show that it is not necessarily true that the set of periodic points of order n is closed.

   (d) Show that, for a continuous $f : X \to X$, if the orders of the periodic points of f are all less than or equal to some fixed number K, then the set $\mathrm{Per}(f)$ of all periodic points of f is closed. You may use without proof the fact that finite unions of closed sets are necessarily closed.

2. Give an example of a set $X \subseteq \mathbb{R}^n$ for some n and a continuous function $f : X \to X$ such that $\mathrm{Per}(f)$ is not closed.

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Question Sheet 2: Solutions

1. *Let $f : X \to X$ be a continuous function on a set $X \subseteq \mathbb{R}$.*

   (a) *Show that the set Fix(f) of all fixed points of f is closed.*

   Let (x_n) be a sequence of fixed points of f with $x_n \to x$. We must show that x is a fixed point of f. Since f is continuous, $f(x_n) \to f(x)$. Therefore $x_n = f(x_n) \to f(x)$ and $x_n \to x$. Hence $f(x) = x$; i.e. x is a fixed point of f.

   (b) *Hence, or otherwise, show that, for each positive integer n, the set Per$_n(f)$ of all periodic points of period n is closed. (You may use without proof the fact that the composition $f \circ g$ of two continuous functions is necessarily continuous.)*

   Since f is continuous and compositions of continuous functions are continuous, the nth iterative power f^n is continuous. Now $\mathrm{Per}_n(f) = \mathrm{Fix}(f^n)$, so the result just proved, applied to f^n in place of f, shows that $\mathrm{Per}_n(f)$ is closed.

   (c) *By considering the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -x$, or otherwise, show that it is not necessarily true that the set of periodic points of order n is closed.*

   For the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -x$, the point 0 is fixed and all other points are periodic of order 2. Thus the set of points periodic of order 2 is $\mathbb{R} \setminus \{0\}$, which is not closed in $\mathbb{R}$.

   (d) *Show that, for a continuous $f : X \to X$, if the orders of the periodic points of f are all less than or equal to some fixed number K, then the set Per(f) of all periodic points of f is closed. You may use without proof the fact that finite unions of closed sets are necessarily closed.*

   We have

   $$\mathrm{Per}(f) = \bigcup_{n=1}^{K} \mathrm{Per}_n(f),$$

   so $\mathrm{Per}(f)$ is a finite union of closed sets and is therefore closed.

   Alternatively, we may observe that

   $$\mathrm{Per}(f) = \mathrm{Per}_{K!}(f),$$

   because $K!$ is a multiple of every number less than or equal to K Therefore $\mathrm{Per}(f)$ is closed.

2. *Give an example of a set $X \subseteq \mathbb{R}^n$ for some n and a continuous function $f : X \to X$ such that Per(f) is not closed.*

   Example 1.7 in the notes (the function $f : S^1 \to S^1$ given by $f(\theta) = 2\,\theta \pmod{2\pi}$) provides a suitable example. We have already observed that $\mathrm{Per}(f)$ is dense in S^1 and that not every point of S^1 is periodic (some, such as $\theta = \pi$ are eventually periodic but not periodic). These two facts imply that $\mathrm{Per}(f)$ is not closed.

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Question Sheet 3

To be handed in on Thursday 22 October.

1. For each of the following functions, use a calculator to iterate the function from a (small) selection of initial values. Use graphical iteration, again from a small selection of initial values, to illustrate the dynamics of the function. Proofs are not required – the idea is to get the 'feel' of graphical iteration. (Remember to have your calculator in *radians* mode.)

   (a) $C(x) = \cos(x)$, (b) $S(x) = \sin(x)$, (c) $E(x) = e^x$, (d) $F(x) = e^{x-1}$, (e) $F(x) = \arctan(x)$.

2. For each of the following functions f, find all the fixed points and classify them as attracting, repelling or non-hyperbolic.

   (a) $f(x) = x^3 - \frac{1}{9}x$;

   (b) $f(x) = x^3 - x$.

3. Prove Theorem (4.6): that if p is a repelling fixed point of a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, (i.e. $|f'(p)| > 1$), then there is an open interval $U = (p - \delta, p + \delta)$ such that, for every $x \in U \setminus \{p\}$, there is a positive integer n such that $f^n(x) \notin U$.

PMA324 Chaos 2009–10
Question Sheet 3: Solutions

1. *For each of the following functions, use a calculator to iterate the function from a (small) selection of initial values. Use graphical iteration, again from a small selection of initial values, to illustrate the dynamics of the function. Proofs are not required. (Remember to have your calculator in* radians *mode.)*

   (a) $C(x) = \cos(x)$, *(b)* $S(x) = \sin(x)$, *(c)* $E(x) = e^x$, *(d)* $F(x) = e^{x-1}$, *(e)* $F(x) = \arctan(x)$.

   The graphical iterations are given on a separate sheet. Note that for the functions C, S and A, the first application of the function produces a value in $[-1, 1]$. Thereafter, in the cases of S and A, subsequent iterations converge to the fixed point 0 from above or from below. In the case of C, the second iteration produces a value in $[0, 1]$; subsequent iterations converge to a fixed point $0.739\ldots$ alternating above and below. The function E has no fixed points: from all initial values, iterates tend to $+\infty$. The function F has a fixed point 1: from initial values less than or equal to 1, iterates converge upwards to 1; from initial values greater than 1, iterates tend to $+\infty$.

2. *For each of the following functions f, find all the fixed points and classify them as attracting, repelling or non-hyperbolic.*

   (a) $f(x) = x^3 - \frac{1}{9}x$;
   Solving $x = x^3 - \frac{1}{9}x$, we find the fixed points at $x = 0, \pm\sqrt{10}/3$. Since $f'(x) = 3x^2 - \frac{1}{9}$, we have: $f'(0) = -\frac{1}{9}$, making 0 attractive; $f'(\pm\sqrt{10}/3) = 29/9$, making the points $\pm\sqrt{10}/3$ both repelling.

   (b) $f(x) = x^3 - x$.
   The fixed points are at $x = 0, \pm\sqrt{2}$, $f'(0) = -1$, making 0 non-hyperbolic, and $f'(\pm\sqrt{2}) = 5$, making the points $\pm\sqrt{2}$ both repelling.

3. *Prove Theorem (4.6): that if p is a repelling fixed point of a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, (i.e. $|f'(p)| > 1$), then there is an open interval $U = (p - \delta, p + \delta)$ such that, for every $x \in U \setminus \{p\}$, there is a positive integer n such that $f^n(x) \notin U$.*

   Suppose that p is a fixed point of f and $|f'(p)| > 1$. Let $k = (1 + |f'(p)|)/2$ and $\varepsilon = (|f'(p)| - 1)/2$.

   Since f' is continuous, $|f'|$ is continuous, so there exists $\delta > 0$ such that

   $$1 < k = |f'(p)| - \varepsilon < |f'(y)| < |f'(p)| + \varepsilon$$

   for all $y \in (p - \delta, p + \delta)$.

   By the Mean Value Theorem, if $0 < |x - p| < \delta$, then there exists y between x and p such that

   $$\frac{f(x) - f(p)}{x - p} = f'(y),$$

   so

   $$\left|\frac{f(x) - f(p)}{x - p}\right| = |f'(y)| > k.$$

   Thus

   $$|f(x) - f(p)| \geqslant k|x - p| \qquad (x \in (p - \delta, p + \delta)),$$

   i.e.

   $$|f(x) - p| \geqslant k|x - p| \qquad (x \in (p - \delta, p + \delta)).$$

   So, if $x \in (p - \delta, p + \delta)$ *and* $|f(x) - p| < \delta$, then

   $$|f^2(x) - p| \geqslant k|f(x) - p| \geqslant k^2|x - p|;$$

et cetera. Thus we have

$$|f^n(x) - p| \geqslant k^n|x - p|,$$

provided that

$$x, f(x), f^2(x), \ldots, f^{n-1}(x) \in (p - \delta, p + \delta).$$

Since $k > 1$, if $x \neq p$ then $k^n|x - p| \to \infty$ as $n \to \infty$. Consequently, there must exist n such that $f^n(x) \notin U$. This is the desired result.

PMA324 Chaos 2009–10
Question Sheet 4

Not to be handed in. Solutions will be posted on Thursday 29 October.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^3 - x^2 - x + 1$, find all the fixed points and classify them as attracting, repelling or non-hyperbolic.

2. By definition, a periodic point p of a function f is hyperbolic if and only if $|(f^m)'(p)| \neq 1$ where m is the order of p for f. Show that this is equivalent to requiring $|(f^M)'(p)| \neq 1$ where M is any period of p for f; i.e. that p be a hyperbolic fixed point of f^M. (Hint: use the 'Note' in lectures about derivatives of iterates.)

3. Show that, if p is a hyperbolic fixed point of a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, then there is an interval $(p - \varepsilon, p + \varepsilon)$ about p which contains no other fixed points of f; i.e. the hyperbolic fixed points are *isolated*. Deduce that if f has no periodic points of order greater than N, for some finite N, then, for every hyperbolic periodic point p of f, there is an interval $(p - \varepsilon, p + \varepsilon)$ about p which contains no other periodic points of f; i.e. the hyperbolic periodic points of f are isolated. (Hint: find a common period for all the periodic points of f and use question 2).

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Question Sheet 4: Solutions

1. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^3 - x^2 - x + 1$, find all the fixed points and classify them as attracting, repelling or non-hyperbolic.*

   To find the fixed points, we solve $f(x) - x = 0$, i.e. the cubic $2x^3 - x^2 - 2x + 1 = 0$. The roots are $x = +1, -1, 1/2$. Then $f'(x) = 6x^2 - 2x - 1$, so $f'(1) = 3$, $f'(-1) = 7$ and $f'(1/2) = -1/2$. Therefore $+1$ and -1 are repelling and $1/2$ is attracting.

2. By definition, a periodic point p of a function f is hyperbolic if and only if $|(f^m)'(p)| \neq 1$ where m is the order of p for f. Show that this is equivalent to requiring $|(f^M)'(p)| \neq 1$ where M is any period of p for f; i.e. that p be a hyperbolic fixed point of f^M. (Hint: use the 'Note' in lectures about derivatives of iterates.)

   Following the 'Note': if p is periodic for f, order m, and $M = km$ is some period, then, by the Chain Rule,

$$\begin{aligned} (f^M)'(p) &= f'(f^{M-1}(p))\,(f^{M-1})'(p) \\ &= f'(f^{M-1}(p))\,f'(f^{M-2}(p))\,(f^{M-2})'(p) \\ \ldots &= f'(f^{M-1}(p))\,f'(f^{M-2}(p))\ldots f'(f(p))\,f'(p) \\ &= \left[f'(p_m)\,f'(p_{m-1})\ldots f'(p_2)\,f'(p_1)\right]^k \\ &= \left[(f^m)'(p)\right]^k, \end{aligned}$$

   where $\{p_1, p_2, \ldots, p_m\}$ is the orbit of p. Hence $|(f^M)'(p)| \neq 1$ if and only if $|(f^m)'(p)| \neq 1$.

3. *Show that, if p is a hyperbolic fixed point of a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, then there is an interval $(p - \varepsilon, p + \varepsilon)$ about p which contains no other fixed points of f; i.e. the hyperbolic fixed points are* isolated.

   Suppose the desired conclusion is false; suppose that for every $\varepsilon > 0$ there is a point $q \in (p-\varepsilon, p+\varepsilon) \setminus \{p\}$ with $f(q) = q$. Then, by the Mean Value Theorem, there exists $x \in (p, q)$, (or $x \in (q, p)$ if $q < p$) such that

$$f'(x) = \frac{f(q) - f(p)}{q - p} = \frac{q - p}{q - p} = 1.$$

   Thus, for all $\varepsilon > 0$, there is a point $x \in (p - \varepsilon, p + \varepsilon)$ with $f'(x) = 1$. Since f' is continuous, it follows that $f'(p) = 1$, contradicting the hypothesis that p is hyperbolic.

   Alternatively, one may argue that if p is hyperbolic, then p is either attracting or repelling. If p is attracting, then there exists $\varepsilon > 0$ such that, for every $x \in (p-\varepsilon, p+\varepsilon)$, $f^n(x) \to p$ as $n \to \infty$. Thus there can be no fixed points in $(p - \varepsilon, p + \varepsilon) \setminus \{p\}$. Likewise, if p is repelling, then there exists $\varepsilon > 0$ such that, for every $x \in (p - \varepsilon, p + \varepsilon) \setminus \{p\}$, $f^n(x) \notin (p - \varepsilon, p + \varepsilon)$ for some n. Thus, again, there can be no fixed points in $(p - \varepsilon, p + \varepsilon) \setminus \{p\}$.

   Deduce that if f has no periodic points of order greater than N, for some finite N, then, for every hyperbolic periodic point p of f, there is an interval $(p - \varepsilon, p + \varepsilon)$ about p which contains no other periodic points of f; i.e. the hyperbolic periodic points of f are isolated. (Hint: find a common period for all the periodic points of f and use question 2).

   If f has no periodic points of order greater than N, then $M = N!$ is a period of all the periodic points of f. Thus, the periodic points of f are just the fixed points of f^M. By question 2 the hyperbolic periodic points of f are just the hyperbolic fixed points of f^M. The result follows by applying the first part of the question to f^M.

PMA324 Chaos 2009–10
Question Sheet 5

Not to be handed in. Solutions will be posted on Thursday 5 November.

1. For $\mu > 0$, let $g_\mu : \mathbb{R} \to \mathbb{R}$ be defined by

$$g_\mu(x) = \frac{\mu}{1 + x^2}.$$

   Show that g_μ has one and only one fixed point p. (You will find that p is given, in terms of μ, by a certain cubic equation. Fortunately, this question can be done without solving that cubic. Alternatively, use MAPLE.)

   Show that p is attracting for $\mu < 2$, non-hyperbolic for $\mu = 2$ and repelling for $\mu > 2$.

2. Show that the value of μ beyond which the 2-cycle for F_μ born at $\mu = 3$ is repelling (i.e. the next bifurcation point beyond 3) is $1 + \sqrt{6}$.

   (This question is messy because it involves computing

$$(F_\mu^2)'(q) = (F_\mu^2)'(r) = F_\mu'(q)F_\mu'(r),$$

   where q, r are two of the roots of the quartic $F_\mu^2(x) = x$, the other two being 0 and $p = (\mu - 1)/\mu$. Since you know two roots of this quartic, it is not too difficult to solve it; but it is neater to get at sums and products of roots and solve the whole problem without ever finding q, r themselves. Alternatively, use MAPLE.)

   What is special about the parameter value $\mu = 1 + \sqrt{5}$?

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Question Sheet 5: Solutions

1. *For $\mu > 0$, let $g_\mu : \mathbb{R} \to \mathbb{R}$ be defined by $g_\mu(x) = \mu/(1+x^2)$. Show that g_μ has one and only one fixed point p. (You will find that p is given, in terms of μ, by a certain cubic equation. Fortunately, this question can be done without solving that cubic.)*

   Show that p is attracting for $\mu < 2$, non-hyperbolic for $\mu = 2$ and repelling for $\mu > 2$.

   Here is the manual solution; a MAPLE solution will be posted on the course web page.

   A point p is a fixed point of g_μ if and only if

$$p = \frac{\mu}{1+p^2},$$

   i.e.

$$p^3 + p = \mu. \qquad (1)$$

   This is a non-trivial cubic and so has at least one real root. To see that it has only one, we observe that $p^3 + p$ is a strictly increasing function of p and so can take the value μ at most once. Note that, since $x^3 + x$ is increasing and is zero and at $x = 0$, we have $\mu > 0 \;\Rightarrow\; p > 0$.

   To test whether p is attracting or repelling, we must look at $|g'_\mu(p)|$. Differentiating g_μ, we obtain

$$g'_\mu(p) = \frac{-2\mu p}{(1+p^2)^2} = \frac{-2p^2}{(1+p^2)},$$

   on substituting for μ from (1). Therefore

$$\begin{aligned} p \text{ is attracting} \quad &\Leftrightarrow \quad \frac{2p^2}{(1+p^2)} < 1 \\ &\Leftrightarrow \quad 2p^2 < (1+p^2) \\ &\Leftrightarrow \quad p < 1, \text{ since } p > 0, \\ &\Leftrightarrow \quad \mu < 2. \end{aligned}$$

   The conditions for p to be repelling and non-hyperbolic follow by changing $<$ to $>$ and $=$, respectively.

2. *Show that the value of μ beyond which the 2-cycle for F_μ born at $\mu = 3$ is repelling (i.e. the next bifurcation point beyond 3) is $1 + \sqrt{6}$.*

   (This question is messy because it involves computing

$$(F_\mu^2)'(q) = (F_\mu^2)'(r) = F'_\mu(q)F'_\mu(r),$$

   where q, r are two of the roots of the quartic $F_\mu^2(x) = x$, the other two being 0 and $p = (\mu-1)/\mu$. Since you know two roots of this quartic, it is not too difficult to solve it; but it is neater to get at sums and products of roots and solve the whole problem without ever finding q, r themselves.)

   What is special about the parameter value $\mu = 1 + \sqrt{5}$?

   Again, this is the manual solution; a MAPLE solution will be posted on the course web page.

   The fixed points of F_μ are 0 and $p = (\mu-1)/\mu$. These are therefore two of the fixed points of F_μ^2. Let the other two be q, r. Then $0, p, q, r$ are the roots of the quartic $F_\mu^2(x) = x$. Writing out this quartic, it becomes

$$\mu^2 x(1-x)(1-\mu x(1-x)) = x,$$

   i.e.

$$\mu^3x^4 - 2\mu^3x^3 + \mu^2(1+\mu)x^2 + (1-\mu^2)x = 0.$$

Therefore the cubic

$$x^3 - 2x^2 + \frac{1+\mu}{\mu}x + \frac{1-\mu^2}{\mu^3} = 0.$$

has roots p, q, r. Therefore $p + q + r = 2$, so $q + r = 2 - p = (\mu + 1)/\mu$, and $pqr = -(1 - \mu^2)/\mu^3$, so $qr = (1 + \mu)/\mu^2$.

By the Chain Rule,

$$\begin{aligned}
(F_\mu^2)'(q) &= F_\mu'(F_\mu(q))F_\mu'(q) \\
&= F_\mu'(r)F_\mu'(q) \\
&= \mu^2(1-2r)(1-2q) \\
&= \mu^2(1-2(r+q)+4rq) \\
&= \mu^2\left(1 - 2\frac{\mu+1}{\mu} + 4\frac{1+\mu}{\mu^2}\right) \\
&= -\mu^2 + 2\mu + 4.
\end{aligned}$$

Elementary calculations show that the quadratic $-\mu^2 + 2\mu + 4$ is a decreasing function of μ for $\mu \geqslant 1$ and that, within this region,

$$-\mu^2 + 2\mu + 4 = \begin{cases} 1 & \text{at } \mu = 3 \\ -1 & \text{at } \mu = 1 + \sqrt{6}. \end{cases}$$

It follows that q, r are attracting periodic points of F_μ for $3 < \mu < 1 + \sqrt{6}$ and repelling periodic points for $\mu > 1 + \sqrt{6}$.

When $\mu = 1 + \sqrt{5}$,

$$(F_\mu^2)'(p) = (F_\mu^2)'(q) = -\mu^2 + 2\mu + 4 = 0.$$

The 2-cycle is superattracting.

PMA324 Chaos 2009–10
Question Sheet 6

To be handed in on Thursday 12 November.

1. Using the Intermediate Value Theorem, show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with a periodic point, then f has a fixed point.

2. **The 'doubling construction'.**
Given a continuous function $f : [0, 1] \to [0, 1]$ we define the *double* F of f by

$$F(x) = \begin{cases} \frac{2}{3} + \frac{1}{3}f(3x) & (0 \leqslant x \leqslant \frac{1}{3}) \\ \alpha x + \beta & (\frac{1}{3} < x < \frac{2}{3}) \\ x - \frac{2}{3} & (\frac{2}{3} \leqslant x \leqslant 1), \end{cases}$$

where the constants α, β are chosen to make F continuous at 1/3 and 2/3.

(a) Draw sketch graphs of a *typical* f (draw a random squiggle) and the corresponding F to illustrate this construction.

(b) Show that F has a fixed point in the interval (1/3,2/3). Call it x_0. By observing that

$$|F(x) - x_0| \geqslant 2|x - x_0| \qquad (x \in (1/3, 2/3)),$$

or otherwise, prove that F has no periodic points in (1/3,2/3) except x_0.

(c) Show that there are no periodic points of odd order in $[0, 1/3] \cup [2/3, 1]$.

(d) Show that if $p \in [0, 1]$ is a periodic point of order n for f, then the points $p/3$ and $(p + 2)/3$ are periodic of order $2n$ for F.

(e) Show that all the periodic points of F in $[0, 1/3] \cup [2/3, 1]$ are of the above form.

To summarize: you have shown that the orders of the periodic points of F are twice the orders of the periodic points of f, together with 1.

3. Find a continuous function $f : [0, 1] \to [0, 1]$ with at least one fixed point, but no periodic points of order greater than 1. Using the doubling construction, deduce that, for every $n \geqslant 0$, there is a continuous function $F_n : [0, 1] \to [0, 1]$ with at least one periodic point of order 2^n, but no periodic points of orders other than $1, 2, 4, 8, \ldots, 2^n$.

PMA324 Chaos 2009–10
Question Sheet 6: Solutions

1. *Using the Intermediate Value Theorem, show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with a periodic point, then f has a fixed point.*

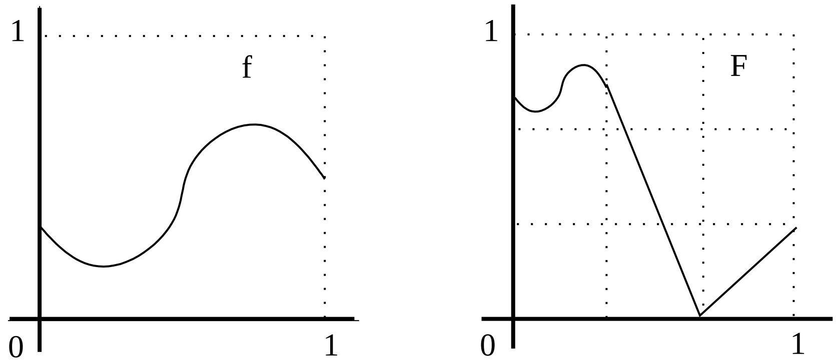
   If $f(x) > x$ for all x, then for every $p \in \mathbb{R}$ the sequence $p, f(p), f^2(p), \ldots$ is strictly increasing, so we cannot have $f^n(p) = p$ with $n \geqslant 1$; p cannot be periodic. Likewise, if $f(x) < x$ for all x, then f has no periodic points. Therefore, if f has a periodic point, then there must exist x, y such that $f(x) \leqslant x$ and $f(y) \leqslant y$; i.e. $f(x) - x \leqslant 0$ and $f(y) - y \geqslant 0$. By the Intermediate Value Theorem, there is a point p between x and y with $f(p) - p = 0$; i.e. p is a fixed point of f.

2. *Given a continuous function $f : [0,1] \to [0,1]$ we define the* double F *of* f *by*

$$F(x) = \begin{cases} \frac{2}{3} + \frac{1}{3}f(3x) & (0 \leqslant x \leqslant \frac{1}{3}) \\ \alpha x + \beta & (\frac{1}{3} < x < \frac{2}{3}) \\ x - \frac{2}{3} & (\frac{2}{3} \leqslant x \leqslant 1), \end{cases}$$

   where the constants α, β are chosen to make F continuous at 1/3 and 2/3.

   (a) *Draw sketch graphs of a typical f and the corresponding F to illustrate this construction.*



   (b) *Show that F has a fixed point in the interval (1/3,2/3). Call it x_0. By observing that*

$$|F(x) - x_0| \geqslant 2|x - x_0| \qquad (x \in (1/3, 2/3)),$$

   or otherwise, prove that F has no periodic points in (1/3,2/3) except x_0.

   The function F is continuous in [1/3,2/3] and

$$F(\frac{1}{3}) - \frac{1}{3} \geqslant \frac{1}{3} > 0 > -\frac{2}{3} = F(\frac{2}{3}) - \frac{2}{3}.$$

   By the Intermediate Value Theorem, $F(x) - x$ has a zero in (1/3,2/3); i.e. F has a fixed point, x_0, say.

   In (1/3,2/3), F has the form $F(x) = \alpha x + \beta$. We have

$$\begin{aligned} \frac{1}{3}\alpha + \beta &= F(\frac{1}{3}) \geqslant \frac{2}{3}, \\ \frac{2}{3}\alpha + \beta &= F(\frac{2}{3}) = 0. \end{aligned}$$

   Hence $\alpha \leqslant -2$. Writing F in the form

$$F(x) = x_0 + \alpha(x - x_0) \qquad (\frac{1}{3} \leqslant x \leqslant \frac{2}{3}),$$

we see that,

$$|F(x) - x_0| = |\alpha|\,|x - x_0| \geqslant 2|x - x_0| \qquad (\frac{1}{3} \leqslant x \leqslant \frac{2}{3}),$$

Repeated application of this shows that, for $x \in (1/3, 2/3)$,

$$|F^n(x) - x_0| \geqslant 2^n |x - x_0|,$$

provided $F(x), \dots, F^{n-1}(x) \in (1/3, 2/3)$. It follows that, for $x \in (1/3, 2/3)\backslash\{x_0\}$, the iterates $F^n(x)$ eventually leave (1/3,2/3). Since F maps $[0, 1/3] \cup [2/3, 1]$ into itself, the iterates can never return to (1/3,2/3) having once left it. Therefore, no such x can be periodic.

(c) *Show that there are no periodic points of odd order in* $[0, 1/3] \cup [2/3, 1]$.

Since $F : [0, 1/3] \to [2/3, 1]$ and $F : [2/3, 1] \to [0, 1/3]$, we see that if n is odd then $F^n : [0, 1/3] \to [2/3, 1]$ and $F^n : [2/3, 1] \to [0, 1/3]$. Therefore F^n has no fixed points in $[0, 1/3] \cup [2/3, 1]$; hence F has no periodic points of odd order in this set.

(d) *Show that if* $p \in [0, 1]$ *is a periodic point of order* n *for* f*, then the points* $p/3$ *and* $(p + 2)/3$ *are periodic of order* $2n$ *for* F.

If $p \in [0, 1]$, then $(p + 2)/3 \in [2/3, 1]$ and

$$\begin{aligned} F((p + 2)/3) &= p/3, \\ F(p/3) &= (f(p) + 2)/3. \end{aligned}$$

Hence

$$\begin{aligned} F^{2k}((p + 2)/3) &= (f^k(p) + 2)/3, \\ F^{2k}(p/3) &= f^k(p)/3. \end{aligned} \tag{1}$$

Thus, if p is periodic of period k for f, then $p/3$ and $(p + 2)/3$ are periodic of **period** $2k$ for F, and if either $p/3$ or $(p + 2)/3$ is periodic of period $2k$ for F, then p is periodic of period k for f.

If, now, p is periodic of **order** n for f, then $p/3$ and $(p+2)/3$ are periodic of **period** $2n$ for F and if either had a smaller order, say $2m < 2n$, (it has to be even, by the previous part), then p would have period $m < n$ for f, contradicting the assumption that n is the order of p.

(e) *Show that all the periodic points of* F *in* $[0, 1/3] \cup [2/3, 1]$ *are of the above form.*

If $x/3 \in [0, 1/3]$ or $(x + 2)/3 \in [2/3, 1]$ is periodic, then its order must be even and so, by (1), x is periodic for f and so these points are of the above form.

3. *Find a continuous function* $f : [0, 1] \to [0, 1]$ *with at least one fixed point, but no periodic points of order greater than 1. Using the doubling construction, deduce that, for every* $n \geqslant 0$*, there is a continuous function* $F_n : [0, 1] \to [0, 1]$ *with at least one periodic point of order* 2^n*, but no periodic points of orders other than* $1, 2, 4, 8, \dots, 2^n$.

The function $f(x) = x$ has every point fixed and therefore no periodic points of higher order.

Defining $F_0 = f$ starts an inductive construction of the desired functions F_n. Given F_n with a periodic point of order 2^n but none of orders other than $1, 2, 4, 8, \dots, 2^n$, we define F_{n+1} to be the double of F_n. Then F_{n+1} has a periodic point of order 2^{n+1}, but none of orders other than $2, 4, 8, \dots, 2^{n+1}$ and 1.

NOTE: in order to get a similar example g with $g : \mathbb{R} \to \mathbb{R}$ it suffices to take f as above and define

$$g(x) = \begin{cases} f(0), & x < 0; \\ f(x), & 0 \leqslant x \leqslant 1; \\ f(1), & x > 1. \end{cases} .$$

Because g maps $\mathbb{R} \to [0, 1]$ and $[0, 1] \to [0, 1]$, iterates of a point outside $[0, 1]$ go into $[0, 1]$ and remain there, so points outside $[0, 1]$ can never be periodic. Inside $[0, 1]$, $g^n(x) = f^n(x)$ for all n, so g has the same periodic points as f, with the same orders. Therefore $g : \mathbb{R} \to \mathbb{R}$ has precisely the same set of orders of periodic points as $f : [0, 1] \to [0, 1]$.

PMA324 Chaos 2009–10
Question Sheet 7

Not to be handed in. Solutions will be posted on Wednesday 25 November.

1. Show that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ with a periodic point of order 7 and no periodic points of order 5. (Hint: make $f(0) = 3$, $f(1) = 6$, $f(2) = 5$, $f(3) = 4$, $f(4) = 2$, $f(5) = 1$, $f(6) = 0$.)

2. Show that there is an everywhere *differentiable* function $f : \mathbb{R} \to \mathbb{R}$ which has a periodic point of order 5, but no periodic points of order 3. You need not define the function explicitly, but you should explain how it is constructed. It will be a smoothed version of the example given in lectures, but you need to specify what properties are preserved in the smoothing in order that the proof still works.

PMA324 Chaos 2009–10
Question Sheet 7: Solutions

1. *Show that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ with a periodic point of order 7 and no periodic points of order 5. (Hint: make $f(0) = 3$, $f(1) = 6$, $f(2) = 5$, $f(3) = 4$, $f(4) = 2$, $f(5) = 1$, $f(6) = 0$.)*

   Let $f : \mathbb{R} \to \mathbb{R}$ be defined as suggested, with f linear between the points $0, 1, \ldots, 6$ and $f(t) = 3$ for $t < 0$, $f(t) = 0$ for $t > 6$. Then 0,1,2,3,4,5,6 are periodic points of f of order 7, and there are no periodic points outside $[0, 6]$, since $f : \mathbb{R} \setminus [0, 6] \to [0, 6]$.

   We may show how intervals are mapped under f by writing:

$$[2,3] \xrightarrow{f} [4,5] \xrightarrow{f} [1,2] \xrightarrow{f} [5,6] \xrightarrow{f} [0,1] \xrightarrow{f} [3,6] \xrightarrow{f} [0,4] \xrightarrow{f} [2,6] \xrightarrow{f} [0,5] \xrightarrow{f} [1,6] \xrightarrow{f} [0,6].$$

   Hence

$$\begin{aligned} f^5([2,3]) &= [3,6], \\ f^5([4,5]) &= [0,4], \\ f^5([1,2]) &= [2,6], \\ f^5([5,6]) &= [0,5], \\ f^5([0,1]) &= [1,6], \end{aligned}$$

   so there are no fixed points for f^5 outside [3,4], except possibly the points 0,1,2,3,4,5,6, but they are periodic points of f of order 7 and so cannot be fixed points for f^5.

   But on [3,4] we have

$$[3,4] \xrightarrow{f} [2,4] \xrightarrow{f} [2,5] \xrightarrow{f} [1,5] \xrightarrow{f} [1,6] \xrightarrow{f} [0,6],$$

   all of these being monotonic decreasing functions. Hence f^5 is monotonic decreasing on [3,4]. Therefore, f^5 has at most one fixed point in that interval, and this must be the fixed point p of f. There are no periodic points of f outside [0,6], since $f(\mathbb{R}) \subseteq [0, 6]$. Thus f has no periodic points of order 5.

2. *Show that there is an everywhere* differentiable *function $f : \mathbb{R} \to \mathbb{R}$ which has a periodic point of order 5, but no periodic points of order 3.*

   We need only modify the example given in the lectures, "rounding off the corners" of the graph, so as to produce a differentiable function f such that $f(x) = 2$ for $x < 0$, $f(0) = 2$, $f(1) = 4$, $f(2) = 3$, $f(3) = 1$, $f(4) = 0$, $f(x) = 0$ for $x > 4$ and f is monotonic on [0,1] and on [1,4]. The proof given for the example treated in the lectures uses only these properties of the function, and so the same proof shows that f has periodic points of order 5 but not of order 3.

   A specific (rather slick) example of such a function is

$$f(x) = \begin{cases} 2 & (x < 0) \\ 3 - \cos \pi x & (0 \leqslant x < 1) \\ 2 + 2\cos \frac{\pi}{3}(x-1) & (1 \leqslant x < 4) \\ 0 & (x \geqslant 4). \end{cases}$$

PMA324 Chaos 2009–10
Question Sheet 8

To be handed in on Wednesday 2 December.

1. Let $f : X \to X$ and $g : Y \to Y$ be topologically conjugate by the homeomorphism $h : X \to Y$. Show that if $p \in X$ is an eventually periodic point for f, then $h(p)$ is an eventually periodic point for g.

2. Show that no two of the following dynamical systems (f_i, X_i) are topologically conjugate.

   (a) $X_1 = \mathbb{R}$, $f_1(x) = x^2$ $(x \in \mathbb{R})$;

   (b) $X_2 = \mathbb{R}$, $f_2(x) = x^3$ $(x \in \mathbb{R})$;

   (c) $X_3 = \mathbb{R}$, $f_3(x) = x^2$ $(x \geqslant 0)$, $f_3(x) = 0$ $(x < 0)$;

   (d) $X_4 = \mathbb{Z}$, $f_4(x) = x^2$ $(x \in \mathbb{Z})$.

3. Show that the dynamical systems $(f, \mathbb{R}^+)$, $(g, \mathbb{R})$ given by $f(x) = x^2$ and $g(x) = 2x$ are topologically conjugate. (Here $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.)

4. For $\lambda, \mu > 0$ we define $H_{\lambda,\mu} : \mathbb{R} \to \mathbb{R}$ by

$$H_{\lambda,\mu}(x) = \mu x(1 - \lambda x^2) \quad (x \in \mathbb{R}).$$

   Show that $(H_{\lambda,\mu}, \mathbb{R})$ is topologically conjugate to $(H_{1,\mu}, \mathbb{R})$, for every $\lambda, \mu > 0$. (Try a homeomorphism of the form $x \mapsto ax$ $(x \in \mathbb{R})$, for a suitable constant a.)

PMA324 Chaos 2009–10
Question Sheet 8: Solutions

1. *Let $f : X \to X$ and $g : Y \to Y$ be topologically conjugate by the homeomorphism $h : X \to Y$. Show that if $p \in X$ is an eventually periodic point for f, then $h(p)$ is an eventually periodic point for g.*

   If p is an eventually periodic point for f, then there exist $0 \leqslant r < s$ such that $f^r(p) = f^s(p)$. Hence

$$g^r(h(p)) = h(f^r(p)) = h(f^s(p)) = g^s(h(p)).$$

   Therefore $h(p)$ is an eventually periodic point for g.

2. *Show that no two of the following dynamical systems (f_i, X_i) are topologically conjugate.*

   (a) $X_1 = \mathbb{R}$, $f_1(x) = x^2$ $(x \in \mathbb{R})$;

   (b) $X_2 = \mathbb{R}$, $f_2(x) = x^3$ $(x \in \mathbb{R})$;

   (c) $X_3 = \mathbb{R}$, $f_3(x) = x^2$ $(x \geqslant 0)$, $f_3(x) = 0$ $(x < 0)$;

   (d) $X_4 = \mathbb{Z}$, $f_4(x) = x^2$ $(x \in \mathbb{Z})$.

First we observe that $\mathbb{Z}$ and $\mathbb{R}$ are not homeomorphic, since, for example, the property 'every convergent sequence is eventually constant' is true in $\mathbb{Z}$ but not in $\mathbb{R}$. Now a topological conjugacy is, amongst other things, a homeomorphism. Therefore there can be no topological conjugacy between (f_4, X_4) and any of the other systems.

Next, we count fixed points: (f_1, X_1) has two (0 and 1), as does (f_3, X_3), but (f_2, X_2) has three (-1, 0 and 1). Therefore, (f_2, X_2) is not topologically conjugate to any of the others.

Finally, we count eventually fixed points: (f_1, X_1) has three (-1, 0 and 1), whilst (f_3, X_3) has infinitely many, since all points $x \leqslant 0$ are eventually fixed. Therefore these two systems can not be topologically conjugate. This completes the proof.

3. *Show that the dynamical systems $(f, \mathbb{R}^+)$, $(g, \mathbb{R})$ given by $f(x) = x^2$ and $g(x) = 2x$ are topologically conjugate. (Here $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.)*

   Let $h : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h(x) = \log x$. Then h is a homeomorphism, since it is a bijection and both the logarithm and its inverse, the exponential function, are continuous. Moreover

$$h(f(x)) = \log(x^2) = 2\log x = g(h(x)),$$

   so h is a topological conjugacy.

4. *For $\lambda, \mu > 0$ we define $H_{\lambda,\mu} : \mathbb{R} \to \mathbb{R}$ by*

$$H_{\lambda,\mu}(x) = \mu x(1 - \lambda x^2) \quad (x \in \mathbb{R}).$$

   Show that $(H_{\lambda,\mu}, \mathbb{R})$ is topologically conjugate to $(H_{1,\mu}, \mathbb{R})$, for every $\lambda, \mu > 0$.

   Let $h(x) = ax$ $(x \in \mathbb{R})$ with $a \neq 0$. Then

$$H_{\lambda,\mu}(h(x)) = \mu a x(1 - \lambda a^2 x^2) = a\mu x(1 - x^2) = h(H_{1,\mu}(x)),$$

   if $a = \pm\lambda^{-1/2}$.

PMA324 Chaos 2009–10
Question Sheet 9

Final homework, not to be handed in. Solutions will be posted on Thursday 3 December.

1. Using (a) the fact that a known cardioidal region forms part of the Mandelbrot set M and (b) the characterization of M in terms of the behaviour of the sequence $(Q_c^n(0))$, decide whether or not each of the following points c lies in M:

$$\frac{3}{16}, \qquad 0.4, \qquad \frac{3i}{2}.$$

   You may use a calculator.

2. Let $c = 3/16$. Find the fixed points of Q_c and classify them as attracting/repelling. By considering inverse images of repelling fixed points, or otherwise, find four distinct points in J_c.

PMA324 Chaos 2009–10
Question Sheet 9: Solutions

1. *Using (a) the fact that a known cardioidal region forms part of the Mandelbrot set M and (b) the characterization of M in terms of the behaviour of the sequence $(Q_c^n(0))$, decide whether or not each of the following points c lies in M:*

$$\frac{3}{16}, \qquad 0.4, \qquad \frac{3i}{2}.$$

You may use a calculator.

The cardioidal region intersects the real axis in the interval $(-3/4, 1, 4)$, so $3/16 \in M$.

We calculate $(Q_c^n(0))$ for $c = 0.4$ and $n = 1, 2, 3, \ldots$, up to the point when $|Q_c^n(0)| > 2$. The sequence goes

0.4,

0.56,

0.7136,

0.90922496,

1.2266900278870016,

1.9047684245174127630928582384O256,

4.02814275103854676289545387224631 (approx.).

Since $|Q_c^7(0)| > 2$, we have $c \notin M$.

We do the same for $c = 3i/2$ and find that

$$Q_c^2(0) = -\frac{9}{4} + \frac{3}{2}i,$$

so

$$|Q_c^2(0)| = \sqrt{\frac{81}{16} + \frac{9}{4}} = \sqrt{\frac{117}{16}} > \sqrt{\frac{64}{16}} = 2.$$

so $c \notin M$.

2. *Let $c = 3/16$. Find the fixed points of Q_c and classify them as attracting/repelling. By considering inverse images of repelling fixed points, or otherwise, find four distinct points in J_c.*

To find the fixed points of Q_c, we solve

$$z^2 + \frac{3}{16} = z,$$

obtaining the two roots $z = 3/4$ and $z = 1/4$. The root $1/4$ is an attracting fixed point, since $|Q_c'(z)| = |2z| < 1$. Similarly, the root $3/4$ is a repelling fixed point, since $|2z| > 1$.

The repelling fixed point $3/4$ is in J_c. So too are iterated inverse images of this point. We find some of these.

Solving $Q_c(z) = 3/4$, we obtain

$$c = \pm\sqrt{\frac{3}{4} - c} = \pm\sqrt{\frac{12}{16} - \frac{3}{16}} = \pm\sqrt{\frac{9}{16}} = \pm\frac{3}{4}.$$

This gives us a second point in J_c, namely $-3/4$.

Solving $Q_c(z) = -3/4$, we obtain

$$c = \pm\sqrt{-\frac{3}{4} - c} = \pm\sqrt{-\frac{12}{16} - \frac{3}{16}} = \pm\sqrt{-\frac{15}{16}} = \pm\frac{\sqrt{15}}{4}\,i.$$

This gives us two further points in J_c, as desired.