PMA443 Fractals June 2008 — Solutions

1. (i) [6 marks: unseen problem] *Let*

$$\begin{aligned} A &= \{(1,y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant 1\}, \\ B &= \{(x,y) \in \mathbb{R}^2 : x, y \geqslant 0 \text{ and } x+y=1\}, \\ C &= \{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 = 1\}. \end{aligned}$$

Write down the values of $\rho(A,B)$, $\rho(B,A)$, $\rho(A,C)$ and $\rho(C,A)$, and hence the Hausdorff distances $d_H(A,B)$ and $d_H(A,C)$.

$$\begin{aligned} \rho(A,B) &= 1/\sqrt{2}, \qquad \rho(B,A) = 1, \\ \rho(A,C) &= \sqrt{2}-1, \qquad \rho(C,A) = 2, \\ d_H(A,B) &= \max\{1/\sqrt{2}, 1\} = 1, \\ d_H(A,C) &= \max\{\sqrt{2}-1, 2\} = 2. \end{aligned}$$

1. (ii) [4 marks: bookwork] *Define the* **Cantor Ternary Set** *and describe a characterization of it in terms of ternary expansions.*

The *Cantor Ternary Set* is the set $C \subseteq \mathbb{R}$ defined as follows. Let

$$\begin{aligned} C_0 &= [0,1], \\ C_1 &= [0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1], \\ C_2 &= [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}] \cup [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1], \\ &\ldots \end{aligned}$$

Then let

$$C = \bigcap_{n=0}^{\infty} C_n.$$

The Cantor set is precisely the set of those points in [0,1] which have a ternary expansion consisting entirely of 0s and 2s.

1. (iii) [6 marks: bookwork + unseen problem] *What is meant by the* **attractor** *of an iterated function system (IFS) $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$ on $\mathbb{R}^N$. Prove that the attractor of an IFS $\mathcal{W}$ consists of a single point if and only if all the w_i $(1 \leqslant i \leqslant M)$ have a common fixed point.*

Given an IFS $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$, there is a unique compact non-empty set such that

$$A = \bigcup_{i=1}^{M} w_i(A).$$

This set$A$ is the attractor of $\mathcal{W}$.

The attractor of an IFS $\mathcal{W}$ consisting of a single point $\{p\}$, means that

$$\bigcup_{i=1}^{M} w_i(\{p\}) = \{p\}.$$

Equivalently

$$\{w_i(p) : 1 \leqslant i \leqslant M\} = \{p\}.$$

This is equivalent to saying that $w_i(p) = p$ for $1 \leqslant i \leqslant M$; i.e. that the w_i have a common fixed point.

1. (iv) [9 marks: unseen problem] *Suppose K is a compact subset of $\mathbb{R}^3$ such that, for all $\varepsilon < 1$, the minimum number $N(\varepsilon)$ of ε-balls needed to cover K satisfies*

$$\frac{3n^2}{\log n} \leqslant N(\varepsilon) < \frac{5n^2}{\log n}$$

where n is the positive integer such that

$$\frac{1}{n+1} < \varepsilon \leqslant \frac{1}{n}.$$

Show that K has Kolmogorov dimension and find its Kolmogorov dimension.

With n and ε related as above, we have:

$$\log(n+1) > \log(1/\varepsilon) \geqslant \log n,$$

so

$$\begin{aligned} \frac{\log 5 + 2\log n - \log\log n}{\log n} &= \frac{1}{\log n}\log\left(\frac{5n^2}{\log n}\right) \\ &> \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \\ &> \frac{1}{\log(n+1)}\log\left(\frac{3n^2}{\log n}\right) \\ &= \frac{\log 3 + 2\log n - \log\log n}{\log n}.\frac{\log n}{\log(n+1)} \end{aligned}$$

As $\varepsilon \to 0$, so $n \to \infty$, we have

$$\log n \to \infty, \qquad \frac{\log n}{\log(n+1)} \to 1, \qquad \frac{\log\log n}{\log n} \to 0,$$

so the far left and far right terms in the above chain of inequalities both tend to 2, so the Kolmogorov dimension exists and is equal to 2.

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2. (i) [4 marks: bookwork definitions] *Explain what is meant by saying that a mapping $f : X \to Y$ between metric spaces is* **Lipschitz**, *and define the* **Lipschitz constant** *$Lip(f)$ of a Lipschitz mapping. What is meant by saying that f is a* **contraction**?

We say f is Lipschitz if there is a constant λ such that

$$d(f(x_1), f(x_2)) \leqslant \lambda d(x_1, x_2) \qquad (x_1, x_2 \in X).$$

The Lipschitz constant $\operatorname{Lip} f$ is defined to be the least λ for which this holds.

A contraction is a Lipschitz mapping with Lipschitz constant strictly less than 1.

2. (ii) [3 marks: unseen problem] *Prove that if $f : X \to Y$ and $g : Y \to Z$ are both Lipschitz mappings between metric spaces, then so is their composition $gf : X \to Z$, and* $\operatorname{Lip} gf \leqslant (\operatorname{Lip} g)(\operatorname{Lip} f)$.

Since f is Lipschitz, we have

$$d(f(x), f(y)) \leqslant (\operatorname{Lip} f) d(x, y) \qquad (x, y \in X).$$

Since g is Lipschitz, we have

$$\begin{aligned} d(g(f(x)), g(f(y))) &\leqslant (\operatorname{Lip} g) d(f(x), f(y)) \\ &\leqslant (\operatorname{Lip} g)(\operatorname{Lip} f) d(x, y) \qquad (x, y \in X) \qquad . \end{aligned}$$

Hence gf is Lipschitz with $\mathrm{Lip}\, gf \leqslant (\mathrm{Lip}\, g)(\mathrm{Lip}\, f)$.

2. (iii) [5 marks: unseen problem] *Let $(X, d_X), (Y, d_Y)$ be (non-empty) metric spaces and define the product metric on $X \times Y$ by*

$$d\left((x_1, y_1), (x_2, y_2)\right) = \max\left\{d_X(x_1, x_2), d_Y(y_1, y_2)\right\}.$$

Prove that the mapping $\pi_X : X \times Y \to X$ defined by $\pi_X(x, y) = x$ is Lipschitz. Is it a contraction? Justify your answer.

We have

$$\begin{aligned} d(\pi_X(x_1, y_1), \pi_X(x_2, y_2)) &= d_X(x_1, x_2) \\ &\leqslant \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \\ &= d((x_1, y_1), (x_2, y_2)) \end{aligned}$$

so π_X is Lipschitz with Lipschitz constant at most 1. In fact, the Lipschitz constant is exactly 1 if X has more than one element, since if $y_1 = y_2$ and $x_1 \neq x_2$ then

$$d(\pi_X(x_1, y_1), \pi_X(x_2, y_2)) = d_X(x_1, x_2) = d((x_1, y_1), (x_2, y_2)),$$

so π_X is not a contraction.

2. (iv)(a) [7 marks: unseen problem] *If $f : X \to Y$ is a function, we define its* graph $\mathrm{Gr}f$ *by*

$$\mathrm{Gr}f = \{(x, y) \in X \times Y : y = f(x)\}.$$

Define $g : X \to \mathrm{Gr}f$ by $g(x) = (x, f(x))$ $(x \in X)$. Prove that if f is Lipschitz then g is biLipschitz and $\mathrm{Lip}\, g \leqslant \max\{\mathrm{Lip}\, f, 1\}$.

Let $\lambda = \mathrm{Lip}\, f$; then

$$d_Y(f(x_1), f(x_2)) \leqslant \lambda d_X(x_1, x_2) \qquad (x_1, x_2 \in X).$$

It follows that

$$\begin{aligned} d(g(x_1), g(x_2)) &= d((x_1, f(x_1)), (x_2, f(x_2))) \\ &= \max\{d_X(x_1, x_2), d_Y(f(x_1), f(x_2))\} \\ &\leqslant \max\{d_X(x_1, x_2), \lambda d_X(x_1, x_2)\} \\ &\leqslant \max\{1, \lambda\} d_X(x_1, x_2). \end{aligned}$$

So g is Lipschitz with $\mathrm{Lip}\, g \leqslant \max\{\mathrm{Lip}\, f, 1\}$.

The mapping g is invertible with inverse $\pi_X : \mathrm{Gr}f \to X$ defined as above: $\pi_X(x, y) = x$, and we have shown that this is Lipschitz. Therefore g is biLipschitz.

2. (iv)(b) [2 marks: unseen problem] *What can you deduce about the Hausdorff dimensions of X and $\mathrm{Gr}f$?*

Since there is a biLipschitz mapping between X and $\mathrm{Gr}f$, their Hausdorff dimensions are the same.

2. (iv)(c) [4 marks: unseen problem] *Prove that if g is Lipschitz, then so is f, with* $\mathrm{Lip}\, f \leqslant \mathrm{Lip}\, g$.

The mapping $\pi_Y : X \times Y \to Y$ defined by $\pi_Y(x, y) = y$ is Lipschitz, by the same argument as in (iv), so if g is Lipschitz, then $f = \pi_Y g$ is Lipschitz by part (ii), with $\mathrm{Lip}\, f \leqslant (\mathrm{Lip}\, \pi_Y)(\mathrm{Lip}\, g) \leqslant 1.\mathrm{Lip}\, g$.

Alternatively, we can argue directly:

$$\begin{aligned} d_Y(f(x_1), f(x_2)) &\leqslant \max\{d_X(x_1, x_2), d_Y(f(x_1), f(x_2))\} \\ &= d((x_1, f(x_1)), (x_2, f(x_2))) \\ &= d(g(x_1), g(x_2)) \\ &\leqslant (\mathrm{Lip}\, g) d_X(x_1, x_2). \end{aligned}$$

so f is Lipschitz with $\mathrm{Lip}\, f \leqslant \mathrm{Lip}\, g$.

3. (i) [6 marks: bookwork] *Let $\mathcal{H}_N$ be the set of all non-empty compact subsets of $\mathbb{R}^N$. Define the quantities $d(x,B)$ and $\rho(A,B)$ for $x \in \mathbb{R}^N$ and $A, B \in \mathcal{H}_N$. Define the* **Hausdorff metric** *d_H on $\mathcal{H}_N$. (You do not need to show that the function you define is a metric.)*

For $x \in \mathbb{R}^N$ and $K \in \mathcal{H}_N$ we define

$$d(x,B) := \inf\{d(x,b) : b \in B\}.$$

Then, for $A, B \in \mathcal{H}_N$

$$\rho(A,B) := \sup\{d(x,B) : x \in A\}.$$

Finally we define the Hausdorff metric by

$$d_H(A,B) = \max\{\rho(A,B), \rho(B,A)\} \quad (A, B \in \mathcal{H}_N).$$

3. (ii) [4 marks: bookwork] *Define the mapping $W : \mathcal{H}_N \to \mathcal{H}_N$ associated with an IFS $\mathcal{W}$ on $\mathbf{R}^N$, proving that it does map $\mathcal{H}_N$ into $\mathcal{H}_N$.*

Given an IFS $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$, we define a map $W : \mathcal{H}_N \to \mathcal{H}_N$ by

$$W(K) = \bigcup_{i=1}^{M} w_i(K) \qquad (K \in \mathcal{H}_N).$$

Since each w_i is continuous, the and K is compact and non-empty, each $w_i(K)$ is compact and non-empty; hence $W(K)$, being a finite union of compact sets, is compact and non-empty.

3. (iii) [2 marks: bookwork] *Let $\mathcal{W} = \{w_0, w_1\}$ be the IFS on $\mathbb{R}$ given by*

$$w_0(x) = x/3, \qquad w_1(x) = (x+2)/3, \qquad (x \in \mathbb{R}).$$

What is the attractor A of this particular IFS?

The attractor is the Cantor ternary set C, since this is compact and non-empty and $W(C) = C$.

3. (iv) [4 marks: unseen problem] *Describe the sets $W^n(K)$ where (a) $K = [0,1]$ and (b) $K = \{0\}$.*

Let

$$\begin{aligned} C_0 &= [0,1], \\ C_1 &= [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], \\ C_2 &= [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1], \\ &\ldots \end{aligned}$$

so that

$$C = \bigcap_{n=0}^{\infty} C_n.$$

Then $W^n([0,1]) = C_n$ and $W^n(\{0\})$ is the set of left-hand end points of the intervals of C_n.

3. (v) [4 marks: unseen problem] *Evaluate $d_H(K, W(K))$ and $d_H(K, A)$ for each of these two choices of K.*

For $K = [0,1]$, we have $W(K) = C_1$, so $\rho((K, W(K)) = 1/6$ and $\rho((W(K), K) = 0$, so $d_H(K, W(K)) = 1/6$, and $\rho(K, A) = 1/6$ and $\rho(A, K) = 0$, so $d_H(K, A) = 1/6$.

For $K = \{0\}$, we have $W(K) = \{0, 2/3\}$, so $\rho((K, W(K)) = 0$ and $\rho((W(K), K) = 2/3$, so $d_H(K, W(K)) = 2/3$, and $\rho(K, A) = 0$ and $\rho(A, K) = 1$, so $d_H(K, A) = 1$.

3. (vi) [5 marks: bookwork + unseen problem] *State* **Barnsley's Collage Theorem** *and verify directly that it holds for the IFS $\mathcal{W}$ and the two choices of starting set K above.*

Barnsley's Collage Theorem: Let $K \in \mathcal{H}_N$ and $\varepsilon > 0$ be given. Let $\mathcal{W}$ be an IFS such that

$$d_H\left(K, \bigcup_{i=1}^{M} w_i(K)\right) < \varepsilon.$$

Let A be the attractor of $\mathcal{W}$. Then

$$d_H(A, K) < \frac{\varepsilon}{1-s},$$

where $s = \max_i \operatorname{Lip} w_i$.

This is the form in which the Collage Theorem is typically applied. An equivalent formulation is that

$$d_H(A, K) \leqslant \frac{1}{1-s} d_H(K, W(K)).$$

In the example given, $s = 1/3$, so $\frac{1}{1-s} = \frac{3}{2}$.
For $K = [0,1]$, we have $d_H(A, K) = 1/6$ and $\frac{1}{1-s} d_H(K, W(K)) = \frac{3}{2}.\frac{1}{6} = \frac{1}{4}$ so the result holds.
For $K = \{0\}$, we have $d_H(A, K) = 1$ and $\frac{1}{1-s} d_H(K, W(K)) = \frac{3}{2}.\frac{2}{3} = 1$ so the result holds (and is sharp).

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4. (i) [3 marks: bookwork definition] *Define the notion of the* **Kolmogorov dimension** *of a non-empty compact subset K of a metric space in terms of the minimum number $N(\varepsilon)$ of ε-balls needed to cover the set.*

We define the **Kolmogorov dimension** of K by

$$\text{Kdim}K = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)},$$

if this limit exists.

4. (ii) [5 marks: bookwork] *Let $A \subseteq K \subseteq B$ be non-empty compact subsets of a metric space X such that both A and B have Kolmogorov dimension and* $\text{Kdim}A = \text{Kdim}B$. *Show that K has Kolmogorov dimension and* $\text{Kdim}K = \text{Kdim}A = \text{Kdim}B$.

Since every covering of B is a covering of K and every covering of K is a covering of A, we have

$$N_A(\varepsilon) \leqslant N_K(\varepsilon) \leqslant N_B(\varepsilon).$$

Therefore

$$\frac{\log N_A(\varepsilon)}{\log(1/\varepsilon)} \leqslant \frac{\log N_K(\varepsilon)}{\log(1/\varepsilon)} \leqslant \frac{\log N_B(\varepsilon)}{\log(1/\varepsilon)}.$$

The outer terms of this inequality both tend to the same limit as $\varepsilon \to 0$, so the middle term tends to the this limit, by the Sandwich Rule. The result follows.

4. (iii) [2 marks: bookwork] *State a characterization of the Kolmogorov dimension of K in terms of the largest number $M = M(\varepsilon)$ for which there is set $\{x_1, x_2, \ldots, x_M\} \subseteq K$ with $d(x_i, x_j) \geqslant \varepsilon$ for all $i \neq j$.*

For a non-empty compact set K in a metric space the limit

$$\lim_{\varepsilon \to 0} \frac{\log M(\varepsilon)}{\log(1/\varepsilon)}$$

exists if and only if $\text{Kdim}K$ exists, in which case they are equal.

4. (iv) [15 marks: bookwork.] *Let $K \subseteq X, L \subseteq Y$ be non-empty compact subsets of metric spaces and suppose that K and L both have Kolmogorov dimension. Show that the subset $K \times L \subseteq X \times Y$ (which you may assume to be non-empty and compact) has Kolmogorov dimension, and*

$$\text{Kdim}(K \times L) = \text{Kdim}K + \text{Kdim}L.$$

Proof. We observe that the definition of the metric on a product means that

$$B((x,y);\varepsilon) = B(x;\varepsilon) \times B(y;\varepsilon) \qquad ((x,y) \in X \times Y).$$

Hence

$$N_{K\times L}(\varepsilon) \leqslant N_K(\varepsilon)N_L(\varepsilon),$$

for if

$$K \subseteq \bigcup_{i=1}^{N} B(x_i;\varepsilon) \qquad \text{and} \qquad L \subseteq \bigcup_{j=1}^{M} B(y_j;\varepsilon),$$

where $N = N_K(\varepsilon), M = N_L(\varepsilon)$, then

$$\begin{aligned} K \times L &\subseteq \bigcup_{i=1}^{N}\bigcup_{j=1}^{M} B(x_i;\varepsilon) \times B(y_j;\varepsilon) \\ &= \bigcup_{i=1}^{N}\bigcup_{j=1}^{M} B((x_i,y_j);\varepsilon), \end{aligned}$$

so $K \times L$ is covered by $N_K(\varepsilon)N_L(\varepsilon)$ ε-balls.

We also have

$$M_{K\times L}(\varepsilon) \geqslant M_K(\varepsilon)M_L(\varepsilon),$$

for if $x_1, \ldots, x_N \in K$ with $d(x_i, x_j) \geqslant \varepsilon$ $(i \neq j)$, and $y_1, \ldots, y_M \in L$ with $d(y_i, y_j) \geqslant \varepsilon$ $(i \neq j)$, then the points (x_i, y_j) $(i = 1, 2, \ldots, N;\ j = 1, 2, \ldots M)$ in $K \times L$ are distance at least ε apart.

Combining these, we obtain:

$$\begin{aligned} \frac{\log M_K(\varepsilon)}{\log(1/\varepsilon)} + \frac{\log M_L(\varepsilon)}{\log(1/\varepsilon)} &= \frac{\log(M_K(\varepsilon)M_L(\varepsilon))}{\log(1/\varepsilon)} \\ &\leqslant \frac{\log M_{K\times L}(\varepsilon)}{\log(1/\varepsilon)} \\ &\leqslant \frac{\log N_{K\times L}(\varepsilon/2)}{\log(1/\varepsilon)} \\ &\leqslant \frac{\log(N_K(\varepsilon/2)N_L(\varepsilon/2))}{\log(1/\varepsilon)} \\ &= \frac{\log N_K(\varepsilon/2)}{\log(1/\varepsilon)} + \frac{\log N_L(\varepsilon/2)}{\log(1/\varepsilon)}. \end{aligned}$$

As $\varepsilon \to 0$, the first of these expressions tends to $\mathrm{Kdim}K + \mathrm{Kdim}L$, and the last also tends to $\mathrm{Kdim}K + \mathrm{Kdim}L$. Hence, by the Sandwich Rule, $\mathrm{Kdim}(K \times L)$ exists and is equal to $\mathrm{Kdim}K + \mathrm{Kdim}L$.

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5. (i) [3 marks: bookwork definition] *Explain what is meant by saying that a subset A of a metric space X is d-***null***, where d is a positive real number.*

The set A is d-null iff, for every $\varepsilon > 0$, there is a covering of A by open balls

$$A \subseteq \bigcup_{i=1}^{\infty} B(x_i;\delta_i) \qquad \text{with} \qquad \sum_{i=1}^{\infty} \delta_i^d < \varepsilon.$$

5. (ii) [5 marks: bookwork] *Prove that if a set A is d-null, then it is d'-null for every $d' \geqslant d$.*

For every $\varepsilon \in (0,1)$, there is a covering of A,

$$A \subseteq \bigcup_{i=1}^{\infty} B(x_i;\delta_i) \qquad \text{with} \qquad \sum_{i=1}^{\infty} \delta_i^d < \varepsilon.$$

Now for each i, $\delta_i^d < \varepsilon < 1$, so $\delta_i < 1$, so $\delta_i^{d'} \leqslant \delta_i^d$. Therefore

$$\sum_{i=1}^{\infty} \delta_i^{d'} < \varepsilon.$$

The fact that we have proved this for $0 < \varepsilon < 1$ is enough to show that A is d'-null, since if it holds for some ε it certainly holds for any larger ε.

5. (iii) [2 marks: bookwork definition] *Hence define the notion of the* **Hausdorff dimension** *Hdim A.*

The *Hausdorff dimension* of A is the number

$$\text{Hdim}A = \inf\{d : A \text{ is } d\text{-null}\}.$$

If there is no d such that A is d-null, we write $\text{Hdim}A = \infty$.

5. (iv) [6 marks: bookwork] *Define the terms* **similitude** *and* **open set condition** *and state* **Hutchinson's theorem** *for the Hausdorff dimension of the attractor of an iterated function system.*

A *similitude* is a mapping $f : \mathbb{R}^N \to \mathbb{R}^N$ (say) such that for some $\lambda \geqslant 0$ we have

$$d(f(x), f(y)) = \lambda d(x,y) \qquad (x, y \in \mathbb{R}^N).$$

The IFS $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$ satisfies the *open set condition* if there exists a non-empty bounded open set $O \subseteq X$ such that

$$\bigcup_{i=1}^{M} w_i(O) \subseteq O$$

and

$$w_i(O) \cap w_j(O) = \emptyset \qquad (i \neq j).$$

Hutchinson's Theorem Let $\mathcal{W}$ be an IFS on $\mathbb{R}^N$ whose contractions are similitudes and which satisfies the open set condition, and let D be its similarity dimension and A its attractor. Then $\text{Hdim}A = D$.

5. (v) [4 marks: unseen problem] *Let* $\mathcal{W} = \{w_0, w_1, w_2\}$ *be the IFS on* $\mathbb{R}$ *given by*

$$w_i(x) = \frac{x+i}{3} \qquad (i = 0,1,2,\ x \in \mathbb{R}).$$

Find the similarity dimension of $\mathcal{W}$ *and identify its attractor.*

This is an IFS consisting of 3 contractions, each with Lipschitz constant $1/3$. Therefore the similarity dimension is

$$\frac{\log 3}{\log 3} = 1.$$

[One can check that the conditions of Hutchinson's Theorem are satisfied and so the Hausdorff dimension of the attractor is 1. However, this is unnecessary if we can easily identify the attractor. We look for a set of Hausdorff dimension 1. It is not hard to guess the answer.]

We claim that the closed unit interval $I = [0,1]$ is the attractor. This is certainly a compact nonempty set; moreover,

$$w_0(I) \cup w_1(I) \cup w_2(I) = [0, \frac{1}{3}] \cup [\frac{1}{3}, \frac{2}{3}] \cup [\frac{2}{3}, 1] = [0,1].$$

The result follows by the uniqueness of the attractor.

5. (vi) [5 marks: unseen problem] *Using Hutchinson's Theorem, find an IFS* $\{w_1, w_2, w_3\}$ *on* $\mathbb{R}$ *whose attractor has Hausdorff dimension 1/2.*

We shall make all the w_i similitudes with the same Lipschitz constant s. For the IFS to have similarity dimension 1/2, we need

$$\frac{1}{2} = \frac{\log 3}{\log 1/s},$$

so $s = 1/9$.

Let w_1, w_2, w_3 be the similitudes

$$w_1(x) = \frac{x}{9}, \qquad w_2(x) = \frac{x+4}{9}, \qquad w_3(x) = \frac{x+8}{9}.$$

We need to check the open set condition. Let $O = (0, 1)$. Then

$$w_1(O) = (0, 1/9), \qquad w_2(O) = (4/9, 5/9), \qquad w_3(O) = (8/9, 1).$$

These are disjoint and contained in O, so the open set condition is satisfied. It follows from Hutchinson's Theorem that the attractor has Hausdorff dimension 1/2.