PMA443 Fractals June 2009 — Solutions

1. (i) [8 marks: bookwork] *Define the* **Cantor Ternary Set** C *and describe a characterization of it in terms of ternary expansions. Deduce that C is uncountable. [You may use the fact that the unit interval* $[0,1]$ *is uncountable.]*

The *Cantor Ternary Set* is the set $C \subseteq \mathbb{R}$ defined as follows. Let

$$\begin{aligned} C_0 &= [0,1], \\ C_1 &= [0,\frac{1}{3}] \cup [\frac{2}{3},1], \\ C_2 &= [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1], \\ &\ldots \end{aligned}$$

Then let

$$C = \bigcap_{n=0}^{\infty} C_n.$$

The Cantor set is precisely the set of those points in [0,1] which have a ternary expansion consisting entirely of 0s and 2s.

We define a map $\theta : C \to [0,1]$ as follows. Let $x \in C$ have a ternary expansion $x = 0.x_1x_2\ldots$ with all the $x_n \in \{0,2\}$, (note that such an expansion is unique), then $\theta(x) \in [0,1]$ is defined by the *binary* expansion

$$0.\frac{x_1}{2}\frac{x_2}{2}\ldots.$$

Now every $y \in [0,1]$ has at least one binary expansion $y = 0.y_1y_2\ldots$ with the $y_i \in \{0,1\}$, so $y = \theta(x)$ where $x = 0.a_1a_2\ldots$, with each $a_i = 2y_i$, is in C. Thus θ is surjective. Since [0,1] is uncountable, it follows that C is uncountable.

1. (ii) [8 marks: part of a homework problem, but with n^s for a general positive integer s replaced by n^2.] *Let*

$$K = \left\{\frac{1}{n^2} : n = 1,2,3,\ldots\right\} \cup \{0\} \subseteq \mathbb{R}.$$

It is known that, for every $\varepsilon \in (0,1/2)$, if $n = n(\varepsilon)$ is the positive integer such that

$$\frac{1}{2(n+1)^3} < \varepsilon \leqslant \frac{1}{2n^3},$$

the least number $N(\varepsilon)$ of ε-balls needed to cover K satisfies

$$n \leqslant N(\varepsilon) \leqslant 2n+1.$$

Deduce that K has Kolmogorov dimension and find its value. Why might this result be seen as a deficiency of Kolmogorov dimension as a measure of dimension?

We have

$$\begin{aligned} \frac{\log n}{\log 2 + 3\log(n+1)} &= \frac{\log n}{\log\left(2(n+1)^3\right)} \\ &\leqslant \frac{\log(N(\varepsilon)}{\log(1/\varepsilon)} \\ &\leqslant \frac{\log(2n+1)}{\log(2n^3)} \\ &= \frac{\log 2 + \log\left(n+\frac{1}{2}\right)}{\log 2 + 3\log n}. \end{aligned}$$

As $\varepsilon \to 0$, we have $n \to \infty$ and the first and last terms both tend to $1/3$, so the Kolmogorov dimension of K exists and is equal to $1/3$.

The set K is countable and we would hope that the dimension of a countable unions of sets would be the sup of their dimensions, in which case the dimension of a countable set would be zero.

1. (iii) [4 marks: bookwork] *Define the* **similarity dimension** *of an iterated function system (IFS) and prove that your definition gives a unique number. Deduce a formula for the similarity dimension of an IFS in which all the contractions have the same Lipschitz constant.*

Let $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$ be an IFS in $\mathbb{R}^N$ with $\operatorname{Lip} w_i = s_i \in (0, 1)$ $(1 \leqslant i \leqslant M)$. The **similarity dimension** of $\mathcal{W}$ is the unique solution D of the equation

$$\sum_{i=1}^{M} s_i^D = 1 \qquad (1)$$

The left hand side of (1) is a continuous, strictly decreasing function of D which is M at $D = 0$ and tends to zero as D tends to infinity; hence there is a unique solution.

If $s_1 = s_2 = \ldots = s_M = s$, then (1) reduces to $Ms^D = 1$, i.e. $\log M + D \log s = 0$,

$$D = \frac{\log M}{\log 1/s}.$$

1. (iv) [5 marks: bookwork definitions] *Explain what is meant by saying that a subset A of a metric space X is d-***null***, where d is a positive real number. Hence define the notion of the* **Hausdorff dimension** *Hdim A.*

The set A is *d-null* iff, for every $\varepsilon > 0$, there is a covering of A by open balls

$$A \subseteq \bigcup_{i=1}^{\infty} B(x_i; \delta_i) \qquad \text{with} \qquad \sum_{i=1}^{\infty} \delta_i^d < \varepsilon.$$

The *Hausdorff dimension* of A is the number

$$\text{Hdim} A = \inf\{d : A \text{ is } d\text{-null}\}.$$

If there is no d such that A is d-null, we write $\text{Hdim} A = \infty$.

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2. (i) [3 marks: bookwork definitions] *Define the terms* **Lipschitz** *and* **biLipschitz** *as applied to mappings between two metric spaces.*

If $f : X \to Y$ is a mapping between metric spaces, then we say f is *Lipschitz* if there is a constant λ such that

$$d(f(x_1), f(x_2)) \leqslant \lambda d(x_1, x_2) \qquad (x_1, x_2 \in X).$$

A mapping $f : X \to Y$ is said to be *biLipschitz* if it is a bijection such that both f and f^{-1} are Lipschitz.

2. (ii) [3 marks: unseen problem] *Prove that if $f : X \to Y$ and $g : Y \to Z$ are both Lipschitz, then so is their composition $gf : X \to Z$, and $\operatorname{Lip} gf \leqslant (\operatorname{Lip} g)(\operatorname{Lip} f)$.*

Since f is Lipschitz, we have

$$d(f(x), f(y)) \leqslant (\operatorname{Lip} f) d(x, y) \qquad (x, y \in X) \qquad .$$

Since g is Lipschitz, we have

$$\begin{aligned} d(g(f(x)), g(f(y))) &\leqslant (\operatorname{Lip} g) d(f(x), f(y)) \\ &\leqslant (\operatorname{Lip} g)(\operatorname{Lip} f) d(x, y) \qquad (x, y \in X). \end{aligned}$$

Hence gf is Lipschitz with $\operatorname{Lip} gf \leqslant (\operatorname{Lip} g)(\operatorname{Lip} f)$.

2. (iii) [5 marks: unseen problem] *Deduce that if $f : X \to Y$ is a biLipschitz map, then*

$$\operatorname{Lip} f \geqslant \frac{1}{\operatorname{Lip} f^{-1}},$$

but that this inequality can be strict. Hint: Let $X = Y = \mathbb{R}^2$ and consider the mapping $f : X \to Y$ defined by

$$f((x, y)) = (x/2, 2y) \qquad ((x, y) \in \mathbb{R}^2).$$

If $I : X \to X$ is the identity mapping, then

$$1 = \operatorname{Lip} I = \operatorname{Lip} f f^{-1} \leqslant \operatorname{Lip} f \operatorname{Lip} \left(f^{-1}\right),$$

so

$$\operatorname{Lip} f \geqslant \frac{1}{\operatorname{Lip} f^{-1}}.$$

Let $X = Y = \mathbb{R}^2$ and consider the mapping $f : X \to Y$ defined by

$$f((x, y)) = (x/2, 2y) \qquad ((x, y) \in \mathbb{R}^2).$$

Then

$$f^{-1}((x, y)) = (2x, y/2) \qquad ((x, y) \in \mathbb{R}^2),$$

so $\operatorname{Lip} f = \operatorname{Lip} f^{-1} = 2$, and

$$\operatorname{Lip} f = 2 > \frac{1}{2} = \frac{1}{\operatorname{Lip} f^{-1}},$$

2. (iv) [4 marks: unseen problem] *Define the terms* **contraction**, **iterated function system** *(IFS) and* **attractor** *of an IFS.*

A *contraction* is a Lipschitz mapping with Lipschitz constant strictly less than 1.

An *iterated function system (IFS)* on $\mathbb{R}^N$ is a finite set $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$ of contractions $w_i : \mathbb{R}^N \to \mathbb{R}^N$.

We say that a set A is *self-similar for $\mathcal{W}$* if

$$A = \bigcup_{i=1}^{M} w_i(A).$$

It may be shown that, given $\mathcal{W}$, there is a unique non-empty compact set A which is self-similar for $\mathcal{W}$. This set is called the *attractor* of the IFS $\mathcal{W}$.

2. (v) [10 marks: unseen problem] *Let $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$ be an IFS on $\mathbb{R}^N$ with attractor A, and let $f : \mathbb{R}^N \to \mathbb{R}^N$ be a biLipschitz mapping such that*

$$\operatorname{Lip} f = \frac{1}{\operatorname{Lip} f^{-1}}.$$

Prove that

$$\mathcal{W}_f = \{f w_1 f^{-1}, f w_2 f^{-1}, \ldots, f w_M f^{-1}\}$$

is an IFS and that its attractor is $f(A)$.

By (ii),

$$\operatorname{Lip} f w_i f^{-1} \leqslant \operatorname{Lip} f \operatorname{Lip} w_i \operatorname{Lip} f^{-1} = \operatorname{Lip} w_i < 1.$$

Thus each $f w_i f^{-1}$ is a contraction, and so $\{f w_1 f^{-1}, f w_2 f^{-1}, \ldots, f w_M f^{-1}\}$ is an IFS.

Given that A is the attractor for $\mathcal{W}$, since f is continuous, $f(A)$ is a compact non-empty set. Further,

$$\begin{aligned} f(A) &= f\left(\bigcup_{i=1}^{M} w_i(A)\right) \\ &= \bigcup_{i=1}^{M} f\left(w_i(A)\right) \\ &= \bigcup_{i=1}^{M} f w_i f^{-1}\left(f(A)\right) \end{aligned}$$

so $f(A)$ is self-similar for $\mathcal{W}_f$. Therefore $f(A)$ is the attractor of $\mathcal{W}_f$.

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3. (i) [8 marks: bookwork] *Let $\mathcal{H}_N$ be the set of all non-empty compact subsets of $\mathbb{R}^N$, where $\mathbb{R}^N$ has the usual metric. Define $d(x, A)$ for $x \in \mathbb{R}^N$, $A \in \mathcal{H}_N$ and show that it is well-defined. Prove that the function $x \mapsto d(x, A)$ is continuous.*

We define

$$d(x, A) = \inf\{d(x, a) : a \in A\},$$

which is well-defined as $A \neq \emptyset$.

Since $a \mapsto d(x, a)$ is a continuous function on the non-empty compact set A, it is bounded and attains its bounds. Therefore

$$d(x, A) = \min\{d(x, a) : a \in A\}.$$

For $x, y \in \mathbb{R}^N$. Let $a_0 \in A$ such that $d(y, A) = d(y, a_0)$. Then

$$d(x, A) \leqslant d(x, a_0) \leqslant d(x, y) + d(y, a_0) = d(x, y) + d(y, A). \tag{2}$$

Likewise

$$d(y, A) \leqslant d(x, y) + d(x, A).$$

Combining these gives

$$|d(x, A) - d(y, A)| \leqslant d(x, y).$$

So, $x \mapsto d(x, A)$ is Lipschitz, with Lipschitz constant 1, and is therefore continuous.

3. (ii) [7 marks: bookwork] *For $A, B \in \mathcal{H}_N$ we define $\rho(A, B) = \sup\{d(x, B) : x \in A\}$. Prove that*

$$d(a, C) \leqslant d(a, B) + \rho(B, C) \qquad (a \in \mathbb{R}^N,\ B, C \in \mathcal{H}_N)$$

and that

$$\rho(A, C) \leqslant \rho(A, B) + \rho(B, C) \qquad (A, B, C \in \mathcal{H}_N).$$

We have

$$\begin{aligned} d(a, C) &\leqslant d(a, b) + d(b, C), \text{ for all } b \in B, \text{ by (2)}, \\ &\leqslant d(a, b) + \rho(B, C), \text{ for all } b \in B. \end{aligned}$$

So

$$\begin{aligned} d(a, C) &\leqslant \inf\{d(a, b) : b \in B\} + \rho(B, C) \\ &= d(a, B) + \rho(B, C). \end{aligned}$$

We observe that

$$\rho(A, B) = \max\{d(x, B) : x \in A\} \qquad (A, B \in \mathcal{H}_N),$$

since $x \mapsto d(x, B)$ is a continuous function on the compact set A . Then

$$\begin{aligned}\rho(A,C) &= \max\{d(a,C) : a \in A\} \\ &= d(a_0, C), \text{ for some } a_0 \in A, \\ &\leqslant d(a_0,B) + \rho(B,C) \\ &\leqslant \rho(A,B) + \rho(B,C).\end{aligned}$$

3. (iii) [2 marks: bookwork] *Define the Hausdorff metric d_H on $\mathcal{H}_N$. (You need not prove that it is a metric.)*

$$d_H(A,B) = \max\{\rho(A,B), \rho(B,A)\} \qquad (A, B \in \mathcal{H}_N).$$

3. (iv) [8 marks: unseen problem]*Let A be a non-empty compact subset of $\mathbb{R}^N$ and, for $x \in \mathbb{R}^N$, define the translation of A by x by*

$$x + A = \{x + a : a \in A\}.$$

Prove that

$$d_H(x + A, A) \leqslant |x| \qquad (x \in \mathbb{R}^N).$$

It suffices to show that

$$\rho(\boldsymbol{x} + A, A) \leqslant |\boldsymbol{x}| \qquad (\boldsymbol{x} \in \mathbb{R}^N),$$

for then

$$\rho(A, \boldsymbol{x} + A) = \rho(-\boldsymbol{x} + (\boldsymbol{x} + A), (\boldsymbol{x} + A)) \leqslant |-\boldsymbol{x}| = |\boldsymbol{x}| \qquad (\boldsymbol{x} \in \mathbb{R}^N),$$

and so

$$d_H(\boldsymbol{x} + A, A) = \max\{\rho(\boldsymbol{x} + A, A), \rho(A, \boldsymbol{x} + A)\} \leqslant |\boldsymbol{x}| \qquad (\boldsymbol{x} \in \mathbb{R}^N).$$

Now $\rho(\boldsymbol{x} + A, A) \leqslant |\boldsymbol{x}|$ is equivalent to $\max\{d(\boldsymbol{x} + \boldsymbol{a}, A) : \boldsymbol{a} \in A\} \leqslant |x|$. But $d(\boldsymbol{x} + \boldsymbol{a}, A) \leqslant d(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{a}) = |\boldsymbol{x}|$, so the result follows.

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4. (i) [3 marks: bookwork definition] *Define the notion of the* **Kolmogorov dimension** *of a non-empty compact subset of a metric space.*

Let $N(\varepsilon)$ be the least number of ε-balls centred on points of K needed to cover K. We define the **Kolmogorov dimension** of K by

$$\mathrm{Kdim} K = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)},$$

if this limit exists.

4. (ii) [12 marks: bookwork] *Prove, from your definition, that the Cantor ternary set C has Kolmogorov dimension* $\log 2/\log 3$.

We recall the definition of C. Let

$$\begin{aligned}C_0 &= [0,1], \\ C_1 &= [0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1], \\ C_2 &= [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}] \cup [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1], \\ &\ldots\end{aligned}$$

Then

$$C = \bigcap_{n=0}^{\infty} C_n.$$

Given $\varepsilon > 0$, let n be such that $3^{-(n+1)} < 2\varepsilon \leqslant 3^{-n}$, i.e. $n = [\log_3(1/2\varepsilon)]$. Because the gaps between the intervals of C_n are all at least 3^{-n}, no ε-ball centred on a point of C can intersect more than one interval of C_n. Now every interval of C_n contains points of C. Therefore, at least 2^n such ε-balls are needed to cover C: i.e. $N(\varepsilon) \geqslant 2^n$.

On the other hand, we can cover C_{n+1} and so C by 2^{n+2} ε-balls centred on the end points of the closed intervals of which C_{n+1} is composed. Therefore $N(\varepsilon) \leqslant 2^{n+2}$.

Thus

$$\frac{n \log 2}{\log 2 + (n+1)\log 3} \leqslant \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \leqslant \frac{(n+2)\log 2}{\log 2 + n \log 3}.$$

As $\varepsilon \to 0$, we have $n \to \infty$, and so the outer terms tend to $(\log 2)/(\log 3)$. The Sandwich Rule implies

$$\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \frac{\log 2}{\log 3},$$

so the Kolmogorov dimension of the Cantor Ternary Set exists, equal to $(\log 2)/(\log 3)$.

4. (iii) [10 marks: unseen problem] *Let E be the set of all mid-points of the 'middle-third' intervals deleted in the construction of C; i.e.*

$$E = \left\{ \frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{18}, \frac{5}{18}, \frac{13}{18}, \frac{17}{18}, \ldots \right\}.$$

Prove that $\mathrm{Kdim}(C \cup E) = \log 2/\log 3$.

Hint: show that if $3^{-(n+1)} < 2\varepsilon \leqslant 3^{-n}$ then the number of ε-balls needed to cover $C_{n+1} \cup E$ is at most

$$2^{n+2} + 2^{n+1} - 1 < 2^{n+3}.$$

We can cover C_{n+1} and so C by 2^{n+2} ε-balls centred on the end points of the closed intervals of which C_{n+1} is composed. This covering leaves a finite number of points of E uncovered. How many? In the construction of C_{n+1}, we removed $2^{n+1} - 1$ middle-thirds intervals, and so there are no more than $2^{n+1} - 1$ points of E remaining to be covered. We cover each of these points by one ε-ball centered on that point. Thus

$$\begin{aligned} N(\varepsilon) &\leqslant 2^{n+2} + 2^{n+1} - 1 \\ &\leqslant 2^{n+3}. \end{aligned}$$

To find $\mathrm{Kdim}(C \cup E)$, we proceed as in (ii) above. We have an upper estimate for $N(\varepsilon)$. Since every covering of E is a covering of C, the lower estimate $N(\varepsilon) \geqslant 2^n$ still holds.

Thus

$$\frac{n \log 2}{\log 2 + (n+1)\log 3} \leqslant \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \leqslant \frac{(n+3)\log 2}{\log 2 + n \log 3}.$$

As $\varepsilon \to 0$, we have $n \to \infty$, and so the outer terms tend to $(\log 2)/(\log 3)$, and the desired result follows as before.

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5. (i) [6 marks: bookwork] *Define the terms* **similitude** *and* **open set condition** *and state* **Hutchinson's theorem** *for the Hausdorff dimension of the attractor of an iterated function system.*

A *similitude* is a mapping $f : \mathbb{R}^N \to \mathbb{R}^N$ (say) such that for some $\lambda \geqslant 0$ we have

$$d(f(x), f(y)) = \lambda d(x, y) \qquad (x, y \in \mathbb{R}^N).$$

The IFS $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$ satisfies the *open set condition* if there exists a non-empty bounded open set $O \subseteq X$ such that

$$\bigcup_{i=1}^{M} w_i(O) \subseteq O$$

and

$$w_i(O) \cap w_j(O) = \emptyset \qquad (i \neq j).$$

Hutchinson's Theorem Let $\mathcal{W}$ be an IFS on $\mathbb{R}^N$ whose contractions are similitudes and which satisfies the open set condition, and let D be its similarity dimension and A its attractor. Then $\mathrm{Hdim}A = D$.

5. (ii) [19 marks: unseen problem] *Let $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}^2$ be given by*

$$x_1 = (0,0), \qquad x_2 = (1,0), \qquad x_3 = (0,1),$$

$$x_4 = (1,1), \qquad x_5 = (0,2).$$

Let w_1, w_2, w_3, w_4, w_5 be affine maps that map points half way to the x_i, that is,

$$w_i(x) = \frac{1}{2}(x + x_i) \qquad (x \in \mathbb{R}^2)$$

for $i = 1, 2, 3, 4, 5$. For each of the following iterated function system, describe the attractor of $\mathcal{W}_i$ (justification is not needed), calculate the similarity dimension of $\mathcal{W}_i$ and say, with a brief justification, whether $\mathcal{W}_i$ satisfies the hypotheses of Hutchinson's Theorem.

$$\begin{aligned}
\mathcal{W}_1 &= \{w_1\}, \\
\mathcal{W}_2 &= \{w_1, w_2\}, \\
\mathcal{W}_3 &= \{w_1, w_2, w_3\}, \\
\mathcal{W}_4 &= \{w_1, w_2, w_3, w_4\}, \\
\mathcal{W}_5 &= \{w_1, w_2, w_3, w_4, w_5\}.
\end{aligned}$$

We note that all of the w_i are similitudes with Lipschitz constant $1/2$. Hence, to check that the hypotheses of Hutchinson's Theorem are satisfied we need only specify a set O that shows that the open set condition is satisfied. Since $\mathcal{W}_n$ has n contractions of Lipschitz constant $1/2$, its similarity dimension D is given by $D = \log n / \log 2$.

1. The attractor of $\mathcal{W}_1$ is the single point x_1. $D = \log 1/\log 2 = 0$. We may take $O = B(x_1; \varepsilon)$ for any $\varepsilon > 0$ to see that the open set condition is satisfied.

   Alternatively, take $O = \{(x, y) : 0 < x, y < 1\}$, and this will do for $\mathcal{W}_1$ to $\mathcal{W}_4$.

2. The attractor of $\mathcal{W}_2$ is the straight line segment between x_1 and x_2; i.e. the set $I = \{(s, 0) : 0 \leqslant s \leqslant 1\}$. $D = \log 2/\log 2 = 1$. The natural choice of open set is $O = \{(x, y) : 0 < x, y < 1\}$ and the open set condition is satisfied.

3. The attractor of $\mathcal{W}_3$ is the Sierpinski Triangle with vertices x_1, x_2, $x_3 D = \log 3/\log 2$. We may take O to be the interior of the triangle $x_1 x_2 x_3$ to see that the open set condition is satisfied.

4. The attractor of $\mathcal{W}_4$ is square $\{(x, y) : 0 \leqslant x, y \leqslant 1\} D = \log 4/\log 2 = 2$. We may take O to be the interior of this square to see that the open set condition is satisfied.

5. The attractor of $\mathcal{W}_5$ is the pentagon $\{(x, y) : 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1, x + y \leqslant 2\}$;$D = \log 5/\log 2$. Since $D > 2$ and the attractor is a subset of $\mathbb{R}^2$, the attractor cannot have dimension D, so the conclusion of Hutchinson's Theorem is false, so the hypotheses are false.