PMA443 Fractals June 2010 — Solutions

1. (i) [4 marks: bookwork + 5 marks: unseen problem similar to homework] *Define the* **Cantor Ternary Set** *C and describe a characterization of it in terms of ternary expansions. Which of the following belong to C: $1/3$, $2/3$, $3/4$, $4/5$, $5/6$? Justify your answer.*

Hint: $1/8 = \sum_{n=1}^{\infty}(1/9)^n$.

The *Cantor Ternary Set* is the set $C \subseteq \mathbb{R}$ defined as follows. Let

$$\begin{aligned} C_0 &= [0,1], \\ C_1 &= [0,\frac{1}{3}] \cup [\frac{2}{3},1], \\ C_2 &= [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1], \\ \cdots \end{aligned}$$

Then let

$$C = \bigcap_{n=0}^{\infty} C_n.$$

The Cantor set is precisely the set of those points in [0,1] which have **a**[1] ternary expansion consisting entirely of 0s and 2s.

The points $1/3, 2/3$ are end-points of intervals of C_i for all $i \geqslant 1$ and are therefore in C. We have $4/5, 5/6 \in (7/9, 8/9)$ and so $4/5, 5/6 \notin C_2$, so $4/5, 5/6 \notin C$.

Alternatively, we can compute the ternary expansions explicitly by doing long divisions in ternary:

$$\begin{aligned} \frac{4}{5} &= 0.210121012101\ldots \\ \frac{5}{6} &= 0.211111111111\ldots \end{aligned}$$

Both these expansions are unique and contain 1s, so $4/5, 5/6 \notin C$.

Since

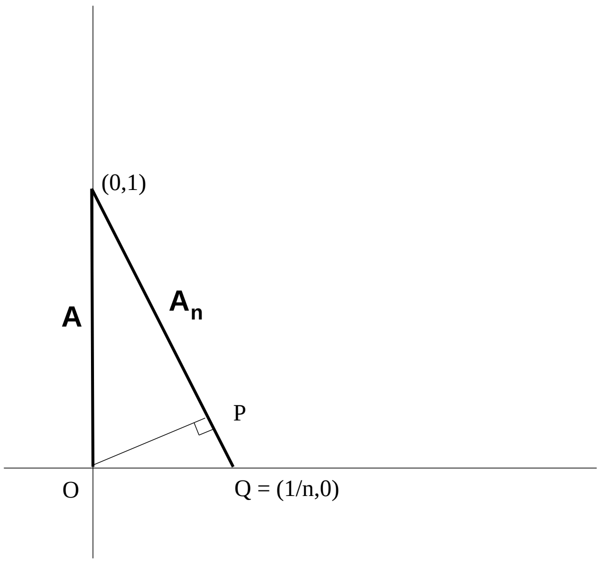
$$\frac{3}{4} = 6\left(\frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \cdots\right),$$

the point $3/4$ has ternary expansion $0.2020202\ldots$. Therefore $3/4 \in C$.

1. (ii) [5 marks: unseen problem] *For $A, B \in \mathcal{H}_N$ we define $\rho(A,B) = \sup\{d(x,B) : x \in A\}$. For $n = 1,2,3,\ldots$ let $A_n \subseteq \mathbb{R}^2$ be the line interval joining the points $(0,1)$ and $(1/n,0)$. Let A be the line interval joining the points $(0,1)$ and $(0,0)$. Write down the values of $\rho(A,A_n)$ and $\rho(A_n,A)$. Prove, from these, that $A_n \to A$ as $n \to \infty$ in the metric space $\mathcal{H}_N$.* We have

$$\rho(A_n,A) = QO = 1/n, \qquad \rho(A,A_n) = OP = 1/\sqrt{1+n^2}.$$

[1] It is important to make clear that some numbers have more than one ternary expansion and these numbers are in C if and only if one of their expansions consists entirely of 0s and 2s.



Therefore

$$d_H(A_n, A) = \max\left\{\frac{1}{n}, \frac{1}{\sqrt{1+n^2}}\right\} = \frac{1}{n}.$$

So $d_H(A_n, A) \to 0$ as $n \to \infty$; i.e. $A_n \to A$ in the metric space $\mathcal{H}_N$.

1. (iii) [7 marks: bookwork + seen problem]

Define the notion of the **Kolmogorov dimension** *of a non-empty compact subset K of a metric space. Prove from your definition that every nonempty finite set has Kolmogorov dimension zero.*

Let $N(\varepsilon)$ be the least number of ε-balls centred on points of K needed to cover K. We define the **Kolmogorov dimension** of K by

$$\mathrm{Kdim}K = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)},$$

if this limit exists.

Let $K = \{x_1, \ldots, x_n\}$ be a nonempty finite set. Then for all $\varepsilon > 0$,

$$K \subseteq \bigcup_{i=1}^{n} B(x_i; \varepsilon).$$

Therefore $N(\varepsilon) \leqslant n$, so

$$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \leqslant \frac{\log n}{\log(1/\varepsilon)} \to 0$$

as $\varepsilon \to 0$. By the Sandwich Rule,

$$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \to 0,$$

i.e. Kdim$K$ exists, equal to zero.

Alternatively, instead of using the fact that $N(\varepsilon) \leqslant n$, we can observe that $N(\varepsilon) = n$ for all $\varepsilon < \min\{d(x_i, x_j) : 1 \leqslant i < j \leqslant n\}$ and then we have

$$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \frac{\log n}{\log(1/\varepsilon)} \to 0,$$

giving the desired result without resort to the Sandwich Rule.

1. (iv) [4 marks: bookwork] *Define the* **similarity dimension** *of an iterated function system (IFS) and prove that your definition gives a unique number. Deduce a formula for the similarity dimension of an IFS in which all the contractions have the same Lipschitz constant.*

Let $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$ be an IFS in $\mathbb{R}^N$ with $\operatorname{Lip} w_i = s_i \in (0, 1)$ $(1 \leqslant i \leqslant M)$. The **similarity dimension** of $\mathcal{W}$ is the unique solution D of the equation

$$\sum_{i=1}^{M} s_i^D = 1 \qquad (1)$$

The left hand side of (1) is a continuous, strictly decreasing function of D which is M at $D = 0$ and tends to zero as D tends to infinity; hence there is a unique solution.

If $s_1 = s_2 = \ldots = s_M = s$, then (1) reduces to $Ms^D = 1$, i.e. $\log M + D \log s = 0$,

$$D = \frac{\log M}{\log 1/s}.$$

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2. (i) [4 marks: bookwork definitions] *Explain what is meant by saying that a mapping $f : X \to Y$ between metric spaces is* **Lipschitz**, *and define the* **Lipschitz constant** *$Lip(f)$ of a Lipschitz mapping. What is meant by saying that f is a* **contraction**?

We say f is Lipschitz if there is a constant λ such that

$$d(f(x_1), f(x_2)) \leqslant \lambda d(x_1, x_2) \qquad (x_1, x_2 \in X).$$

The Lipschitz constant $\operatorname{Lip} f$ is defined to be the least λ for which this holds.

A contraction is a Lipschitz mapping with Lipschitz constant strictly less than 1.

2. (ii) [5 marks: seen problem] *Show that a non-constant mapping $f : X \to Y$ between metric spaces X and Y is Lipschitz with Lipschitz constant less than or equal to λ if and only if $f(B(x; \varepsilon)) \subseteq B(f(x); \lambda\varepsilon)$ for all $x \in X$ and $\varepsilon > 0$.*

The hypothesis that f be non-constant ensures that f cannot be Lipschitz with Lipschitz constant zero; in the other direction, the statement $f(B(x; \varepsilon)) \subseteq B(f(x); \lambda\varepsilon)$ cannot hold if $\lambda = 0$, as that would make the right-hand side empty. Thus we are working throughout with $\lambda > 0$.

The statement

$$f(B(x; \varepsilon)) \subseteq B(f(x); \lambda\varepsilon)$$

is equivalent to

$$f(y) \in f(B(x;\varepsilon)) \quad \Rightarrow \quad f(y) \in B(f(x);\lambda\varepsilon),$$

i.e.

$$y \in B(x;\varepsilon) \quad \Rightarrow \quad f(y) \in B(f(x);\lambda\varepsilon),$$

i.e.

$$d(x,y) < \varepsilon \quad \Rightarrow \quad d(f(x),f(y)) < \lambda\varepsilon. \tag{2}$$

This follows immediately from

$$d(f(x),f(y)) \leqslant \lambda d(x,y) \qquad (x,y \in X), \tag{3}$$

since $\lambda > 0$, and conversely, if (3) were false for some x, y, we could take ε with

$$d(f(x),f(y)) > \lambda\varepsilon > \lambda d(x,y),$$

again using $\lambda > 0$, contradicting (2). This proves the desired equivalence.

2. (iii) [7 marks: unseen problem] *Let $f : [0,\infty) \to [0,\infty)$ be defined by*

$$f(x) = \frac{x}{x+1} \qquad (x \in [0,\infty)).$$

Prove that

$$d(f(x),f(y)) < d(x,y) \qquad (x,y \in [0,\infty), x \neq y),$$

where d is the usual metric on $[0,\infty)$. Is f Lipschitz? If so, what it the Lipschitz constant of f? Is f a contraction? **continue part***Justify your answers.*

(a) For $x, y \in [0,\infty)$ we have

$$\begin{aligned} d(f(x),f(y)) &= |f(x)-f(y)| \\ &= \left|\frac{x}{x+1} - \frac{y}{y+1}\right| \\ &= \left|\frac{x-y}{(x+1)(y+1)}\right| \\ &= \frac{1}{(x+1)(y+1)} d(x,y) \\ &< d(x,y) \end{aligned}$$

when $x \neq y$, since at least one of x, y is strictly positive.

Alternatively, we can use the Mean Value Theorem:

$$\begin{aligned} d(f(x),f(y)) &= |f(x)-f(y)| \\ &= |f'(z)|.|x-y| \text{ for some } z \in (x,y) \subseteq (0,\infty) \\ &= \left|\frac{1}{(z+1)^2}\right|.|x-y| \\ &< 1.|x-y| \text{ since } z > 0 \\ &= d(x,y). \end{aligned}$$

(b) In particular,

$$d(f(x), f(y)) \leqslant 1.d(x, y) \qquad (x, y \in [0, \infty), x \neq y),$$

so f is Lipschitz with Lipschitz constant at most 1.

(c) In fact

$$\text{Lip}\, f = \sup\left\{\frac{d(f(x), f(y))}{d(x, y)} : x \neq y\right\} = \sup\left\{\frac{1}{(x+1)(y+1)} : x \neq y\right\} = 1.$$

(d) Since Lip f is not strictly less than one, F is not a contraction.

2. (iv) [3 marks: bookwork] *What is meant by the* **attractor** *of an iterated function system (IFS)* $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$ *on* $\mathbb{R}^N$.

Given an IFS $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$, there is a unique compact non-empty set such that

$$A = \bigcup_{i=1}^{M} w_i(A).$$

This set$A$ is the attractor of $\mathcal{W}$.

2. (v) [6 marks: unseen problem] *Find an IFS whose attractor is the triangle*

$$T = \{(x, y) \in \mathbb{R}^2 : x \geqslant 0,\ y \geqslant 0 \text{ and } x + y \leqslant 1\}$$

and justify your answer.

Let $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$ where

$$\begin{aligned} w_1(x, y) &= (x/2, y/2) \\ w_2(x, y) &= (x/2, (y+1)/2) \\ w_3(x, y) &= ((x+1)/2, y/2) \\ w_4(x, y) &= ((1-x)/2, (1-y)/2). \end{aligned}$$

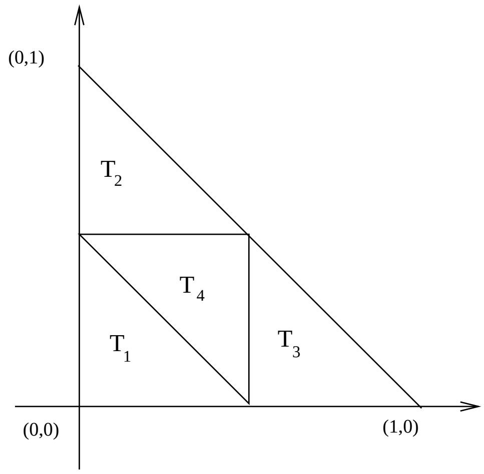
Then we have $w_i(T) = T_i$ and $T = T_1 \cup T_2 \cup T_3 \cup T_4$, so T is self-similar for $\mathcal{W}$. Since T is compact and non-empty, it must be the attractor of $\mathcal{W}$.

[Notice that if w_4 were omitted, we should have an IFS whose attractor would be the Sierpinski gasket with vertices (0,0), (1,0) and (0,1).]

An alternative solution involving only three contractions is $\mathcal{W} = \{w_1, w_2, w_3\}$ where

$$\begin{aligned} w_1(x, y) &= (3x/4, 3y/4) \\ w_2(x, y) &= (3x/4, (3y+1)/4) \\ w_3(x, y) &= ((3x+1)/4, 3y/4). \end{aligned}$$

In this case, the triangles $w_i(T)$ overlap enough to cover T and, as before, it follows that T is the attractor.



3. (i) [4 marks: bookwork] *Define the notion of the* **grid dimension** *of a non-empty compact subset of* $\mathbb{R}^n$

We define the *grid dimension* of a non-empty compact set $K \subseteq \mathbb{R}^n$ as follows. For each $\varepsilon > 0$, we choose a grid of n orthogonal sets of parallel hyperplanes with separation ε. We let $N_g(\varepsilon)$ denote the number of (closed) grid cubes containing points of K and then define

$$\text{griddim} K = \lim_{\varepsilon \to 0} \frac{\log N_g(\varepsilon)}{\log(1/\varepsilon)},$$

if the limit exists.

3. (ii) [21 marks: bookwork] *Prove that the grid dimension of a non-empty compact subset of a* $\mathbb{R}^n$ *exists if and only if the Kolmogorov dimension exists and that it is independent of the positioning of the grids. If you use a subsidiary lemma about limits, such as the 'Comparison Lemma', this should be proved.*

Let C be a closed ε-grid cube containing at least one point $x \in K$. Then the diameter of C, the distance from one vertex to the opposite vertex, is $\sqrt{n}\,\varepsilon$, so $C \subseteq B(x; 2\sqrt{n}\,\varepsilon)$ (we allow a spare factor of 2 here to allow for the case when x is a vertex, the cube being closed and the ball open). Thus every closed ε-grid cube which meets K is contained in a $2\sqrt{n}\,\varepsilon$-ball centred on a point of K. Now K is covered by $N_g(\varepsilon)$ such cubes, and therefore K is covered by $N_g(\varepsilon)$ of these $2\sqrt{n}\,\varepsilon$-balls centred on points of K. Therefore the least number of such balls needed to cover K is no more than $N_g(\varepsilon)$. That is, $N(2\sqrt{n}\,\varepsilon) \leqslant N_g(\varepsilon)$, for all $\varepsilon > 0$: equivalently

$$N(\varepsilon) \leqslant N_g\left(\frac{1}{2\sqrt{n}}\varepsilon\right),$$

for all $\varepsilon > 0$ (by replacing ε by $\varepsilon/(2\sqrt{n})$.

Conversely, every open ball of radius ε meets no more than 3^n ε-grid cubes, so $N_g(\varepsilon) \leqslant 3^n N(\varepsilon)$.

We complete the proof with a technical lemma.

The Comparison Lemma

Let $A(\varepsilon)$, $B(\varepsilon)$ be two positive-real-valued functions on $\mathbb{R}^+$ and suppose that there exist positive constants $\lambda_1, \lambda_2, \mu_1, \mu_2$ such that for all $\varepsilon > 0$

(a) $A(\varepsilon) \leqslant \lambda_1 B(\mu_1 \varepsilon)$ and

(b) $B(\varepsilon) \leqslant \lambda_2 A(\mu_2 \varepsilon)$,

then the limit

$$\lim_{\varepsilon \to 0} \frac{\log A(\varepsilon)}{\log(1/\varepsilon)}$$

exists if and only if the limit

$$\lim_{\varepsilon \to 0} \frac{\log B(\varepsilon)}{\log(1/\varepsilon)}$$

exists, in which case the two limits are equal.

The lemma will complete the proof of our result by putting $A(\varepsilon) = N(\varepsilon)$, $B(\varepsilon) = N_g(\varepsilon)$, $\lambda_1 = 1$, $\mu_1 = 1/(2\sqrt{n})$, $\lambda_2 = 3^n$, and $\mu_2 = 1$.

Thus we have shown that the grid dimension, no matter how the grid is oriented, is equal to the Kolmogorov dimension; so the grid dimension is independent of orientation.

Proof of Lemma. From (b) we have, on replacing ε by $\mu_2 \varepsilon$,

$$\lambda_3 B(\mu_3 \varepsilon) \leqslant A(\varepsilon),$$

where $\lambda_3 = \lambda_2^{-1}$ and $\mu_3 = \mu_2^{-1}$.

Then

$$\begin{aligned}
\left(\frac{\log \lambda_3 + \log B(\mu_3 \varepsilon)}{\log(1/\mu_3 \varepsilon)}\right)\left(\frac{\log(1/\varepsilon) - \log \mu_3}{\log(1/\varepsilon)}\right) &= \frac{\log\left(\lambda_3 B(\mu_3 \varepsilon)\right)}{\log(1/\varepsilon)} \\
&\leqslant \frac{\log A(\varepsilon)}{\log(1/\varepsilon)} \\
&\leqslant \frac{\log\left(\lambda_1 B(\mu_1 \varepsilon)\right)}{\log(1/\varepsilon)} \\
&= \left(\frac{\log \lambda_1 + \log B(\mu_1 \varepsilon)}{\log(1/\mu_1 \varepsilon)}\right)\left(\frac{\log(1/\varepsilon) - \log \mu_1}{\log(1/\varepsilon)}\right).
\end{aligned}$$

Now as $\varepsilon \to 0$,we have $\log(1/\varepsilon) \to \infty$ and so

$$\frac{\log(1/\varepsilon) - \log \mu_i}{\log(1/\varepsilon)} \to 1 \qquad (i = 1, 3),$$

and

$$\frac{\log \lambda_i}{\log(1/\mu_i \varepsilon)} \to 0 \qquad (i = 1, 3).$$

Suppose

$$L_B := \lim_{\varepsilon \to 0} \frac{\log B(\varepsilon)}{\log(1/\varepsilon)}$$

exists; then, as $\varepsilon \to 0$, we have $\mu_i\varepsilon \to 0$, so

$$\frac{\log B(\mu_i\varepsilon)}{\log(1/\mu_i\varepsilon)} \to L_B.$$

Hence, in the above chain of inequalities, the first and last expressions both tend to L_B. Therefore, by the Sandwich Rule,

$$\frac{\log A(\varepsilon)}{\log(1/\varepsilon)} \to L_B,$$

as desired. This proves half of the lemma, but the other half is similar, with the rôles of A and B being reversed.

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4. (i) [6 marks: bookwork] *Explain what is meant by saying that a subset A of a metric space is d-***null,** *where d is a positive real number and define the notion of the* **Hausdorff dimension** Hdim$A$ *of A.*

Let $d > 0$. A subset A of a metric space X is *d-null* if, for every $\varepsilon > 0$, there is a covering of A by open balls

$$A \subseteq \bigcup_{i=1}^{\infty} B(x_i; \delta_i)$$

with

$$\sum_{i=1}^{\infty} \delta_i^d < \varepsilon.$$

Let A be a non-empty (subset of a) metric space. The *Hausdorff dimension* of A is the number

$$\mathrm{Hdim}A = \inf\{d : A \text{ is } d\text{-null}\}.$$

If there is no d such that A is d-null, we write $\mathrm{Hdim}A = \infty$. (If A is d-null for every $d > 0$, the definition gives $\mathrm{Hdim}A = 0$.)

4. (ii)(a) [4 marks: bookwork set as homework] *Let A and B be nonempty subsets of a metric space X. Show that if $A \subseteq B$ then* Hdim$A \leqslant$ Hdim$B$. *Any property of the notion of d-null that you use must be proved.*

We must show that, for any $d > 0$, if B is d-null then so is A, for then

$$\{d : B \text{ is } d\text{-null}\} \subseteq \{d : A \text{ is } d\text{-null}\},$$

so

$$\mathrm{Hdim}B = \inf\{d : B \text{ is } d\text{-null}\} \geqslant \inf\{d : A \text{ is } d\text{-null}\} = \mathrm{Hdim}A.$$

This is immediate since, if we have a covering

$$\bigcup_{i=1}^{\infty} B(x_i; \delta_i)$$

of B with

$$\sum_{i=1}^{\infty} \delta_i^d < \varepsilon,$$

then this is also a covering of A.

4. (ii)(b) [4 marks: unseen problem] *Let A and B be nonempty subsets of a metric space X. Show that*

$$\text{Hdim}(A \cap B) \leqslant \min\{\text{Hdim}A, \text{Hdim}B\}$$

if $A \cap B \neq \emptyset$. Any property of the notion of d-null that you use must be proved.

Since $A \cap B \subseteq A$, (ii)(a) shows that $\text{Hdim}(A \cap B) \leqslant \text{Hdim}A$. Likewise, $\text{Hdim}(A \cap B) \leqslant \text{Hdim}B$. Combining these gives the result.

4. (ii)(c) [7 marks: simplification of homework problem] *Let A and B be nonempty subsets of a metric space X. Show that*

$$\text{Hdim}(A \cup B) = \max\{\text{Hdim}A, \text{Hdim}B\}.$$

Any property of the notion of d-null that you use must be proved.

Suppose A and B are d-null. Given $\varepsilon > 0$, there exist coverings

$$A \subseteq \bigcup_{i=1}^{\infty} B(x_i; \delta_i), \qquad B \subseteq \bigcup_{j=1}^{\infty} B(y_j; \eta_j),$$

such that

$$\sum_{i=1}^{\infty} \delta_i^d < \varepsilon/2 \quad \text{and} \quad \sum_{j=1}^{\infty} \eta_j^d < \varepsilon/2.$$

Then

$$A \cup B \subseteq \bigcup_{i=1}^{\infty} B(x_i; \delta_i) \cup \bigcup_{j=1}^{\infty} B(y_j; \eta_j)$$

and

$$\sum_{i=1}^{\infty} \delta_i^d + \sum_{j=1}^{\infty} \eta_j^d < \varepsilon.$$

Thus $A \cup B$ is d-null.

It follows that

$$\{d : A \cup B \text{ is } d\text{-null}\} \supseteq \{d : A \text{ is } d\text{-null}\} \cap \{d : B \text{ is } d\text{-null}\}.$$

Therefore,

$$\text{Hdim}(A \cup B) \leqslant \max\{\text{Hdim}A, \text{Hdim}B\}.$$

The reverse inequality follows from (ii)(a) because $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

4. (iii) [4 marks: unseen problem] *Give an example showing that the inequality in (ii)(b) can be strict.*

In the metric space $\mathbb{R}$ with the usual metric, let $A = [0,1]$, $B = [1,2]$. Then $\mathrm{Hdim}A = \mathrm{Hdim}B = 1$ but $A \cap B = \{1\}$, so $\mathrm{Hdim}(A \cap B) = 0$.

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5. (i) [6 marks: bookwork] *Define the terms* **similitude** *and* **open set condition** *and state* **Hutchinson's theorem** *for the Hausdorff dimension of the attractor of an iterated function system.*

A *similitude* is a mapping $f : \mathbb{R}^N \to \mathbb{R}^N$ (say) such that for some $\lambda \geqslant 0$ we have

$$d(f(x), f(y)) = \lambda d(x,y) \qquad (x, y \in \mathbb{R}^N).$$

The IFS $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$ satisfies the *open set condition* if there exists a non-empty bounded open set $O \subseteq X$ such that

$$\bigcup_{i=1}^{M} w_i(O) \subseteq O$$

and

$$w_i(O) \cap w_j(O) = \emptyset \qquad (i \neq j).$$

Hutchinson's Theorem Let $\mathcal{W}$ be an IFS on $\mathbb{R}^N$ whose contractions are similitudes and which satisfies the open set condition, and let D be its similarity dimension and A its attractor. Then $\mathrm{Hdim}A = D$.

5. (ii) [12 marks: bookwork] *Let* $\mathcal{W} = \{w_1, w_2, \ldots, w_M\}$ *be an IFS on* $\mathbb{R}^N$ *with attractor* A *and similarity dimension* D. *Show that* $\mathrm{Hdim}A \leqslant D$.

The set A is compact and so bounded; i.e. $A \subseteq B(x;\rho)$ for some $x \in \mathbb{R}^N$ and $\rho > 0$. If $\mathrm{Lip}\, w_i = s_i$ $(1 \leqslant i \leqslant M)$, then $w_i(B(x;\rho)) \subseteq B(w_i(x); s_i\rho)$. Therefore

$$\begin{aligned} A &= \bigcup_{i=1}^{M} w_i(A) \\ &\subseteq \bigcup_{i=1}^{M} w_i(B(x;\rho)) \\ &\subseteq \bigcup_{i=1}^{M} B(w_i(x); s_i\rho). \end{aligned}$$

Repeating the argument:

$$\begin{aligned} A &= \bigcup_{j=1}^{M} w_j(A) \\ &\subseteq \bigcup_{j=1}^{M}\bigcup_{i=1}^{M} w_j(B(w_i(x); s_i\rho)) \\ &\subseteq \bigcup_{i,j=1}^{M} B(w_j w_i(x); s_j s_i\rho). \end{aligned}$$

Generally,

$$A \subseteq \bigcup_{i_1,\ldots,i_n=1}^{M} B(w_{i_1}\ldots w_{i_n}(x); s_{i_1}\ldots s_{i_n}\rho). \tag{4}$$

Now, for every $d > D$, $\sum_{i=1}^{M} s_i^d < 1$, so

$$\sum_{i_1,\ldots,i_n=1}^{M} (s_{i_1}\ldots s_{i_n}\rho)^d = \rho^d \left(\sum_{i=1}^{M} s_i^d\right)^n$$

tends to zero as $n \to \infty$. Thus A is d-null for all $d > D$; so $\mathrm{Hdim} A \leqslant D$.

5. (iii) [7 marks: bookwork] *Let $\mathcal{W} = \{w_1, w_2\}$ be an IFS on $\mathbb{R}^2$ where*

$$w_1(x,y) = \left(\frac{x}{2}, \frac{x}{2}\right), \quad w_2(x,y) = \left(1 - \frac{x}{2}, \frac{x}{2}\right) \qquad ((x,y) \in \mathbb{R}^2).$$

Find the similarity dimension of $\mathcal{W}$. Verify that the attractor of $\mathcal{W}$ is the union of the line segments $(0,0)$ to $(1/2,1/2)$ and $(1/2,1/2)$ to $(1,0)$. What is the Hausdorff dimension of the attractor? If it is not equal to the similarity dimension, explain why Hutchinson's Theorem does not apply to this IFS.

Note. There was an error in the question as originally set. The contraction w_2 should be defined as

$$w_2(x,y) = \left(1 - \frac{x}{2}, \frac{x}{2}\right), \text{ not } w_2(x,y) = \left(1 - \frac{y}{2}, \frac{y}{2}\right).$$

We have $\mathrm{Lip}\, w_1 = \mathrm{Lip}\, w_2 = 1/\sqrt{2}$ so

$$D = \frac{\log 2}{\log \sqrt{2}} = 2.$$

[The question does not require a proof that $\mathrm{Lip}\, w_1 = \mathrm{Lip}\, w_2 = 1/\sqrt{2}$, but because it is easy to think that the Lipschitz constants are the same as for the map

$$(x,y) \mapsto \left(\frac{x}{2}, \frac{y}{2}\right),$$

namely $1/2$, here is a proof. First, we have

$$\begin{aligned} d(w_1(x,y), w_1(x',y')) &= d\left(\left(\frac{x}{2}, \frac{x}{2}\right), \left(\frac{x'}{2}, \frac{x'}{2}\right)\right) \\ &= \sqrt{(\left(\frac{x}{2} - \frac{x'}{2}\right)^2 + \left(\frac{x}{2} - \frac{x'}{2}\right)^2)} \\ &= \frac{1}{\sqrt{2}}|x - x'| \\ &\leqslant \frac{1}{\sqrt{2}} d((x,y),(x',y')) \end{aligned}$$

This shows that w_1 is Lipschitz with Lipschitz constant at most $1/\sqrt{2}$. However, we observe that $d((0,0),(1,0)) = 1$, but

$$d(w_1(0,0), w_1(1,0)) = d((0,0), \left(\frac{1}{2}, \frac{1}{2}\right)) = \frac{1}{\sqrt{2}},$$

so $\operatorname{Lip} w_1 \geqslant 1/\sqrt{2}$. The proof for w_2 is similar.]

Let A be the union of the line segments $(0,0)$ to $(1/2,1/2)$ and $(3/4,1/4)$ to $(1,0)$. Then A is the attractor of $\mathcal{W}$, because A is compact and nonempty and $A = w_1(A) \cup w_2(A)$. As A is a finite union of line segments, $\mathrm{Hdim} A = 1$.

Hutchinson's Theorem does not apply because neither w_1 nor w_2 are similitudes. (The Open Set Condition is satisfied.)