PMA443 Fractals 2009–10
Question Sheet 1

To be handed in on Thursday 18 February.

1. For each of the following subsets of the real line (with the usual metric), state (i) whether or not it is **open**, and (ii) whether or not it is **closed**:

   (a) $[0, 1] \cup \{2\}$;

   (b) $(0, 1) \cup \{2\}$;

   (c) $\mathbb{R} \setminus \{1, 2, 3\}$;

   (d) $\{-1/n : n = 1, 2, 3, \ldots\} \cup [0, 1]$;

   (e) $(0, +\infty)$.

2. Let X, Y be two metric spaces. Prove that a sequence (x_n, y_n) in $X \times Y$ converges to a point (x, y) in the product metric if and only if $x_n \to x$ in X and $y_n \to y$ in Y. Deduce that the projection maps

$$\pi_X : (x, y) \mapsto x : X \times Y \to X,$$

$$\pi_Y : (x, y) \mapsto y : X \times Y \to Y$$

   are continuous.

3. The *product* of two metric spaces (X_1, d_1) and (X_2, d_2) is the space

$$X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$$

   with the metric

$$d((a_1, a_2), (b_1, b_2)) = \max\{d_1(a_1, b_1), d_2(a_2, b_2)\}.$$

   Two alternative metrics on $X_1 \times X_2$ are the "taxi-cab metric"

$$d'((a_1, a_2), (b_1, b_2)) = d_1(a_1, b_1) + d_2(a_2, b_2)$$

   and

$$d''((a_1, a_2), (b_1, b_2)) = \sqrt{d_1(a_1, b_1)^2 + d_2(a_2, b_2)^2}.$$

   Show that, for $a, b \in X_1 \times X_2$,

$$d(a, b) \leqslant d''(a, b) \leqslant d'(a, b) \leqslant 2d(a, b).$$

PMA443 Fractals 2009–10
Question Sheet 1: Solutions

1. *For each of the following subsets of the real line (with the usual metric), state (i) whether or not it is* **open**, *and (ii) whether or not it is* **closed**:

   (a) $[0, 1] \cup \{2\}$;

   (b) $(0, 1) \cup \{2\}$;

   (c) $\mathbb{R} \setminus \{1, 2, 3\}$;

   (d) $\{-1/n : n = 1, 2, 3, \ldots\} \cup [0, 1]$;

   (e) $(0, +\infty)$.

   (a) $[0, 1] \cup \{2\}$ is closed, being the union of two closed sets, but not open, since no open ball centred on 0, 1 or 2 is contained in the set.

   (b) $(0, 1) \cup \{2\}$ is not closed, since the sequence $(1/n)_{n=1}^{\infty}$ is in the set but converges to a point, namely 0, outside the set. It is not open since no open ball centred on 2 is contained in the set.

   (c) $\mathbb{R} \setminus \{1, 2, 3\}$ is open since it is the complement of a closed set. It is not closed since the sequence $(1 - 1/n)_{n=1}^{\infty}$ is in the set but converges to a point, namely 1, outside the set.

   (d) $\{-1/n : n = 1, 2, 3, \ldots\} \cup [0, 1]$ is closed, being the union of the two closed sets $\{-1/n : n = 1, 2, 3, \ldots\} \cup \{0\}$ and $[0, 1]$, but it is not open since no open ball centred on any of the points -1/n, 0, 1 is contained in the set.

   (e) $(0, +\infty)$ is open (to get $B(x, \varepsilon) \subseteq (0, +\infty)$ take $\varepsilon = x$), but it is not closed, since $1/n \to 0 \notin (0, +\infty)$.

2. *Let X, Y be two metric spaces. Prove that a sequence (x_n, y_n) in $X \times Y$ converges to a point (x, y) in the product metric if and only if $x_n \to x$ in X and $y_n \to y$ in Y. Deduce that the projection maps $\pi_X : (x, y) \mapsto x : X \times Y \to X$, $\pi_Y : (x, y) \mapsto y : X \times Y \to Y$ are continuous.*

   We write d for the metric in X, Y and $X \times Y$ and it will be clear from the context which is meant.

   Now $(x_n, y_n) \to (x, y)$ iff $d((x_n, y_n), (x, y)) \to 0$; that is,

   $$\max\{d(x_n, x), d(y_n, y)\} \to 0.$$

   This is equivalent to

   $$d(x_n, x) \to 0 \text{ and } d(y_n, y) \to 0,$$

   i.e.

   $$x_n \to x \text{ and } y_n \to y.$$

To show that π_X is continuous, we must show that

$$(x_n, y_n) \to (x, y) \quad \Rightarrow \quad \pi_X((x_n, y_n)) \to \pi_X((x, y)),$$

i.e.

$$(x_n, y_n) \to (x, y) \quad \Rightarrow \quad x_n \to x,$$

which is part of what we have just proved. The case of π_Y is similar.

3. *The* product *of two metric spaces* (X_1, d_1) *and* (X_2, d_2) *is the space*

$$X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$$

with the metric

$$d((a_1, a_2), (b_1, b_2)) = \max\{d_1(a_1, b_1), d_2(a_2, b_2)\}.$$

Two alternative metrics on $X_1 \times X_2$ *are the "taxi-cab metric"*

$$d'((a_1, a_2), (b_1, b_2)) = d_1(a_1, b_1) + d_2(a_2, b_2)$$

and

$$d''((a_1, a_2), (b_1, b_2)) = \sqrt{d_1(a_1, b_1)^2 + d_2(a_2, b_2)^2}.$$

Show that, for $a, b \in X_1 \times X_2$*,*

$$d(a, b) \leqslant d''(a, b) \leqslant d'(a, b) \leqslant 2d(a, b).$$

We have

$$d_1(a_1, b_1) = \sqrt{d_1(a_1, b_1)^2} \leqslant \sqrt{d_1(a_1, b_1)^2 + d_2(a_2, b_2)^2} = d''(a, b)$$

and likewise

$$d_2(a_2, b_2) \leqslant d''(a, b),$$

so

$$d(a, b) = \max\{d_1(a_1, b_1), d_2(a_2, b_2)\} \leqslant d''(a, b).$$

Then

$$\begin{aligned} d''(a, b) &= \sqrt{d_1(a_1, b_1)^2 + d_2(a_2, b_2)^2} \\ &\leqslant \sqrt{d_1(a_1, b_1)^2 + 2d_1(a_1, b_1)d_2(a_2, b_2) + d_2(a_2, b_2)^2} \\ &= d_1(a_1, b_1) + d_2(a_2, b_2) \\ &= d'(a, b). \end{aligned}$$

Finally,

$$d_1(a_1, b_1) \leqslant \max\{d_1(a_1, b_1), d_2(a_2, b_2)\} = d(a, b)$$

and likewise

$$d_2(a_2, b_2) \leqslant= d(a, b),$$

so, adding,

$$d'((a_1, a_2), (b_1, b_2)) = d_1(a_1, b_1) + d_2(a_2, b_2) \leqslant 2d(a, b).$$

PMA443 Fractals 2009–10
Question Sheet 2

Not to be handed. Solutions will be posted on Thursday 25 February.

1. A real number is said to be **algebraic** if it is a root of an equation

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0,$$

with integer coefficients $a_n, \ldots, a_0$, (not all zero). By considering the size of the set A_N of all numbers x satisfying such an equation with $|a_n| + \ldots + |a_0| \leqslant N$ and $n \leqslant N$, or otherwise, show that the set of all algebraic numbers is countable.

A real number is **transcendental** if it is not algebraic. Show that the set of all transcendental numbers is uncountable. Deduce that transcendental numbers exist! (Actually, this is the easiest way of proving the existence of transcendental numbers. It is much harder to produce a specific transcendental number and very much harder to prove that interesting numbers such as e and π are transcendental.)

2. For each of the following numbers, give its ternary expansion and say whether or not it belongs to the Cantor Ternary Set.

$$\frac{7}{27}, \quad \frac{10}{27}, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}.$$

PMA443 Fractals 2009–10
Question Sheet 2: Solutions

1. *A real number is said to be* **algebraic** *if it is a root of an equation*

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0,$$

with integer coefficients $a_n, \ldots, a_0$, (not all zero). By considering the size of the set A_N of all numbers x satisfying such an equation with $|a_n| + \ldots + |a_0| \leqslant N$ and $n \leqslant N$, or otherwise, show that the set of all algebraic numbers is countable.

A real number is **transcendental** *if it is not algebraic. Show that the set of all transcendental numbers is uncountable. Deduce that transcendental numbers exist!*

For each N, there are only finitely many sequences $(a_0, a_1, \ldots, a_n)$ of integers satisfying $|a_n| + \ldots + |a_0| \leqslant N$ and $n \leqslant N$. For each such sequence, the corresponding equation has no more than n solutions. Therefore A_N is finite. The set A of all algebraic numbers is the union $\bigcup_{N=1}^{\infty} A_N$, and is therefore countable.

The set T of all transcendental numbers is just $\mathbb{R} \setminus A$. We have shown that A is countable; if T were also countable (or empty), then $\mathbb{R} = T \cup A$ would be countable, and we know this to be false. Therefore T must be uncountable and, in particular, non-empty.

2. *For each of the following numbers, give its ternary expansion and say whether or not it belongs to the Cantor Ternary Set.*

$$\frac{7}{27}, \quad \frac{10}{27}, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}.$$

$$\begin{aligned}
\frac{7}{27} &= 0.021 = 0.020222222222\ldots \in C \\
\frac{10}{27} &= 0.101 \notin C \\
\frac{1}{2} &= 0.1111111111\ldots \notin C \\
\frac{1}{3} &= 0.1 = 0.022222222222\ldots \in C \\
\frac{1}{4} &= 0.02020202\ldots \in C \\
\frac{1}{5} &= 0.0121012101210121\ldots \notin C
\end{aligned}$$

PMA443 Fractals 2009–10
Question Sheet 3

Solutions to questions 3, 4, and 5 to be handed in on Thursday 4 March.

1. Show that every decreasing sequence $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ of compact non-empty sets in a metric space has a non-empty intersection.

2. Show that if $K_1 \subseteq X_1$ and $K_2 \subseteq X_2$ are compact subsets of metric spaces X_1, X_2, then $K_1 \times K_2$ is a compact subset of the metric space $X_1 \times X_2$.

3. Show that if $K \subseteq X$ is totally bounded, the x_i in the definition may be taken to lie anywhere in X. (Hint: use the x_i in X for $\varepsilon/2$ to get the desired x_i in K for ε.)

4. Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable then f is Lipschitz with $\operatorname{Lip} f \leqslant \lambda$ if and only if $|f'(x)| \leqslant \lambda$ for all $x \in \mathbb{R}$.

5. Show that a non-constant mapping $f : X \to Y$ between metric spaces X and Y is Lipschitz with Lipschitz constant less than or equal to λ if and only if

$$f(B(x, \varepsilon)) \subseteq B(f(x), \lambda\varepsilon)$$

   for all $x \in X$ and $\varepsilon > 0$.

6. Show that total boundedness and completeness are each preserved by biLipschitz maps. That is, show that if $f : X \to Y$ is biLipschitz, then

   (a) if $K \subseteq X$ is totally bounded, so is $f(K) \subseteq Y$;

   (b) if X is complete, so is Y.

   By considering the map $x \mapsto \tan x : (-\pi/2, \pi/2) \to \mathbb{R}$, show that neither is preserved by homeomorphisms. (You may assume that tan and arctan are continuous on the domains in question.)

PMA443 Fractals 2009–10
Question Sheet 3: Solutions

1. *Show that every decreasing sequence $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ of compact non-empty sets in a metric space has a non-empty intersection.*

   Let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ be a decreasing sequence of compact non-empty sets. For every n, choose $x_n \in K_n$ arbitrarily. Then the sequence (x_n) in the compact set K_1 has a convergent subsequence $x_{n_i} \to x \in K_1$. For each m, we have $(x_{n_i}) \in K_m$ for all sufficiently large i. Removing the initial terms which are not in K_m does not affect the limit, and because K_m is closed, the limit of a sequence in K_m must lie in K_m. Therefore $x \in K_m$. This holds for all m, so $x \in \bigcap_{m=1}^{\infty} K_m$. Thus $\bigcap_{m=1}^{\infty} K_m \neq \emptyset$.

2. *Show that if $K_1 \subseteq X_1$ and $K_2 \subseteq X_2$ are compact subsets of metric spaces X_1, X_2, then $K_1 \times K_2$ is a compact subset of the metric space $X_1 \times X_2$.*

   Let (x_n, y_n) $(n = 1, 2, 3, \dots)$ be a sequence in $K_1 \times K_2$. Then x_n $(n = 1, 2, 3, \dots)$ is a sequence in the compact set K_1. Therefore, it has a convergent subsequence $x_{n_i} \to x$, say. Then y_{n_i} $(i = 1, 2, 3, \dots)$ is a sequence in the compact set K_2. Therefore (y_{n_i}) has a convergent subsequence which, to avoid triple subscripts, we shall designate simply $y_{m_j} \to y$. The sequence $(x_{m_j}, y_{m_j}) \to (x, y)$ in $K_1 \times K_2$ is a convergent subsequence of the original sequence, as desired.

3. *Show that if $K \subseteq X$ is totally bounded, the x_i in the definition may be taken to lie anywhere in X. (Hint: use the x_i in X for $\varepsilon/2$ to get the desired x_i in K for ε.)*

   Suppose K satisfies the definition of total boundedness without the restriction that the centres of the ε-balls lie in K. Then, given $\varepsilon > 0$, there is a finite set $\{y_1, y_2, \dots, y_n\} \subseteq X$ such that

   $$K \subseteq B(y_1, \varepsilon/2) \cup B(y_2, \varepsilon/2) \cup \dots \cup B(y_n, \varepsilon/2).$$

   By removing any redundant balls from this covering, if necessary, we may assume that for each i there is a point $x_i \in K \cap B(y_i, \varepsilon/2)$. By the triangle inequality, $B(y_i, \varepsilon/2) \subseteq B(x_i, \varepsilon)$, so

   $$K \subseteq B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \dots \cup B(x_n, \varepsilon),$$

   as desired.

4. *Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable then f is Lipschitz with* $\operatorname{Lip} f \leqslant \lambda$ *if and only if $|f'(x)| \leqslant \lambda$ for all $x \in \mathbb{R}$.*

   If $\operatorname{Lip} f \leqslant \lambda$, then, for all $x, x_0 \in \mathbb{R}$ with $x \neq x_0$,

   $$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| = \frac{d(f(x), f(x_0))}{d(x, x_0)} \leqslant \lambda$$

so

$$|f'(x_0)| = \left| \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right] \right| \leqslant \lambda.$$

Conversely, if $|f'(x)| \leqslant \lambda$ for all $x \in \mathbb{R}$, then, for all $x \neq y$ in $\mathbb{R}$,

$$\begin{aligned} \frac{d(f(x), f(y))}{d(x, y)} &= \left| \frac{f(x) - f(y)}{x - y} \right| \\ &= |f'(\xi)|, \end{aligned}$$

for some ξ between x and y, by the Mean Value Theorem. Thus

$$\frac{d(f(x), f(y))}{d(x, y)} \leqslant \lambda,$$

so $\operatorname{Lip} f \leqslant \lambda$.

5. *Show that a non-constant mapping $f : X \to Y$ between metric spaces X and Y is Lipschitz with Lipschitz constant less than or equal to λ if and only if $f(B(x, \varepsilon)) \subseteq B(f(x), \lambda\varepsilon)$ for all $x \in X$ and $\varepsilon > 0$.*

   (The hypothesis that f be non-constant is needed to ensure that f cannot be Lipschitz with Lipschitz constant zero; so we are working with $\lambda > 0$.) The statement

$$f(B(x, \varepsilon)) \subseteq B(f(x), \lambda\varepsilon)$$

   is equivalent to

$$f(y) \in f(B(x, \varepsilon)) \quad \Rightarrow \quad f(y) \in B(f(x), \lambda\varepsilon),$$

   i.e.

$$y \in B(x, \varepsilon) \quad \Rightarrow \quad f(y) \in B(f(x), \lambda\varepsilon),$$

   i.e.

$$d(x, y) < \varepsilon \quad \Rightarrow \quad d(f(x), f(y)) < \lambda\varepsilon. \tag{1}$$

   This follows immediately from

$$d(f(x), f(y)) \leqslant \lambda d(x, y) \qquad (x, y \in X), \tag{2}$$

   and conversely, if (2) were false for some $x, y,$, we could take ε with

$$d(f(x), f(y)) > \lambda\varepsilon > \lambda d(x, y),$$

   contradicting (1). This proves the desired equivalence.

6. *Show that total boundedness and completeness are each preserved by biLipschitz maps. That is, show that if $f : X \to Y$ is biLipschitz, then (a) if $K \subseteq X$ is totally bounded, so is $f(K) \subseteq Y$; and (b) if X is complete, so is Y. By considering the map $x \mapsto \tan x : (-\pi/2, \pi/2) \to \mathbb{R}$, show that neither is preserved by homeomorphisms. (You may assume that tan and arctan are continuous on the domains in question.)*

(a) Let $f : X \to Y$ be a biLipschitz map between metric spaces X, Y, with $\mathrm{Lip}\,(f) = \lambda$, $\mathrm{Lip}\,(f^{-1}) = \Lambda$. Suppose $K \subseteq X$ is totally bounded. We show that $f(K) \subseteq Y$ is totally bounded. Given $\varepsilon > 0$, we have

$$K \subseteq B(x_1, \varepsilon/\lambda) \cup \ldots \cup B(x_n, \varepsilon/\lambda)$$

for some $x_1, \ldots x_n \in K$. Now

$$d(f(x_i), f(x)) \leqslant \lambda d(x_i, x)$$

implies

$$x \in B(x_i, \varepsilon/\lambda) \quad \Rightarrow \quad f(x) \in B(f(x_i), \varepsilon).$$

Therefore

$$f(B(x_i, \varepsilon/\lambda) \subseteq B(f(x_i), \varepsilon).$$

Hence

$$\begin{aligned} f(K) &\subseteq f\left(B(x_1, \varepsilon/\lambda) \cup \ldots \cup B(x_n, \varepsilon/\lambda)\right) \\ &= f\left(B(x_1, \varepsilon/\lambda)\right) \cup \ldots \cup f\left(B(x_n, \varepsilon/\lambda)\right) \\ &\subseteq B(f(x_1), \varepsilon) \cup \ldots \cup B(f(x_n), \varepsilon). \end{aligned}$$

Thus $f(K)$ is totally bounded.

(b) Suppose X is complete. To show that Y is complete, we suppose that (y_n) is Cauchy in Y. Then $y_n = f(x_n)$ $(n = 1, 2, 3, \ldots)$ for some sequence $(x_n) \in X$. We show that (x_n) is Cauchy. From that fact that (y_n) is Cauchy, we know that, for all $\varepsilon > 0$ there exists N such that for all $p, q \geqslant N$ we have $d(y_p, y_q) < \varepsilon/\Lambda$. Then, for all $p, q \geqslant N$,

$$d(x_p, x_q) = d(f^{-1}(y_p), f^{-1}(y_q)) \leqslant \Lambda d(y_p, y_q) < \varepsilon.$$

Therefore (x_n) is Cauchy in X. Since X is complete, $x_n \to x$ for some $x \in X$. Therefore $y_n = f(x_n) \to f(x)$, because f is Lipschitz and therefore continuous. Thus Y is complete.

Now let $X = (-\pi/2, \pi/2)$, $Y = \mathbb{R}$ and $f(x) = \tan x$. Then f is a homeomorphism (and $f^{-1} : Y \to X$ is also a homeomorphism), but Y is complete and X is not (the sequence $x_n = \frac{\pi}{2} - \frac{1}{n}$ is Cauchy but not convergent). Moreover, X is totally bounded, but Y is not.

PMA443 Fractals 2009–10
Question Sheet 4

Not to be handed in. Solutions will be posted on Thursday 11 March.

1. Let $x \in \mathbb{R}$ and $A, B \subseteq \mathbb{R}$ be given by

$$x = 0, \qquad A = [0, 2], \qquad B = [1, 4].$$

   State the values of $d(x, A)$, $d(x, B)$, $\rho(A, B)$, $\rho(B, A)$, $d(A, B)$.

2. Let $A, B, C, D \subseteq \mathbb{R}^2$ be given by

$$\begin{aligned} A &= \{(x, y) : -1 \leqslant x \leqslant 1 \text{ and } -1 \leqslant y \leqslant 1\} \\ B &= \{(x, y) : x^2 + y^2 \leqslant 1\} \\ C &= \{(x, y) : x^2 + y^2 = 1\} \\ D &= \{(x, y) \in A : |x| = 1 \text{ or } |y| = 1\} \end{aligned}$$

   Find the Hausdorff distances $d(A, B)$, $d(B, C)$ and $d(C, D)$.

3. Show that it is NOT generally true that

$$d(x, A) = d_H(\{x\}, A) \qquad (x \in \mathbb{R}^N; A \in \mathcal{H}_N).$$

4. Prove that $d(x, A) \leqslant d(x, B) + d_H(B, A)$ $(x \in X; A, B \in \mathcal{H}_N)$.

5. Show that, for $A, B, C \in \mathcal{H}_N$,

$$\rho(A \cup B, C) = \max\{\rho(A, C), \rho(B, C)\}$$

   and

$$\rho(A, B \cup C) \leqslant \min\{\rho(A, B), \rho(A, C)\}.$$

   Deduce that, for $A, B, C, D \in \mathcal{H}_N$,

$$d_H(A \cup B, C \cup D) \leqslant \max\{d_H(A, C), d_H(B, D)\}.$$

PMA443 Fractals 2009–10
Question Sheet 4: Solutions

1. *Let $x \in \mathbb{R}$ and $A, B \subseteq \mathbb{R}$ be given by*

$$x = 0, \qquad A = [0, 2], \qquad B = [1, 4].$$

State the values of $d(x, A)$, $d(x, B)$, $\rho(A, B)$, $\rho(B, A)$, $d(A, B)$.

$$\begin{aligned} d(x, A) &= 0, \text{ since } x \in A; \\ d(x, B) &= 1; \\ \rho(A, B) &= 1; \\ \rho(B, A) &= 2; \\ d(A, B) &= 2. \end{aligned}$$

2. *Let $A, B, C, D \subseteq \mathbb{R}^2$ be given by*

$$\begin{aligned} A &= \{(x, y) : -1 \leqslant x \leqslant 1 \textit{ and } -1 \leqslant y \leqslant 1\} \\ B &= \{(x, y) : x^2 + y^2 \leqslant 1\} \\ C &= \{(x, y) : x^2 + y^2 = 1\} \\ D &= \{(x, y) \in A : |x| = 1 \textit{ or } |y| = 1\} \end{aligned}$$

Find the Hausdorff distances $d(A, B)$, $d(B, C)$ and $d(C, D)$.

$$\rho(A, B) = \sqrt{2} - 1 \text{ and } \rho(B, A) = 0, \text{ so } d(A, B) = \sqrt{2} - 1;$$
$$\rho(B, C) = 1 \text{ and } \rho(C, B) = 0, \text{ so } d(B, C) = 1;$$
$$\rho(C, D) = (\sqrt{2} - 1)/\sqrt{2} \text{ and } \rho(D, C) = \sqrt{2} - 1, \text{ so } d(C, D) = \sqrt{2} - 1.$$

3. *Show that it is NOT generally true that*

$$d(x, A) = d_H(\{x\}, A) \qquad (x \in \mathbb{R}^N; A \in \mathcal{H}_N).$$

We have
$$\rho(\{x\}, A) = \max\{d(y, A) : y \in \{x\}\} = d(x, A)$$
but
$$\rho(A, \{x\}) = \max\{d(a, \{x\}) : a \in A\} = \max\{d(a, x) : a \in A\},$$
which is always greater than or equal to $d(x, A) := \min\{d(a, x) : a \in A\}$ and is generally strictly greater than it. Therefore $d_H(\{x\}, A) = \max\{d(a, x) : a \in A\}$ and this is generally strictly greater than $d(x, A)$. A simple example, in $\mathbb{R}$, is provided by $x = 0$, $A = [1, 2]$, where $d(x, A) = 1$ but $d_H(\{x\}, A) = 2$.

4. *Prove that* $d(x, A) \leqslant d(x, B) + d_H(B, A)$ $(x \in X; A, B \in \mathcal{H}_N)$.

We have $d(x, B) = d(x, b)$ for some $b \in B$. Therefore, with this same b,

$$d_H(B, A) \geqslant \rho(B, A) \geqslant d(b, A) = d(b, a),$$

for some $a \in A$. Then

$$d(x, A) \leqslant d(x, a) \leqslant d(x, b) + d(b, a) \leqslant d(x, B) + d_H(B, A).$$

5. *Show that, for* $A, B, C \in \mathcal{H}_N$,

$$\rho(A \cup B, C) = \max\{\rho(A, C), \rho(B, C)\}$$

and

$$\rho(A, B \cup C) \leqslant \min\{\rho(A, B), \rho(A, C)\}.$$

Deduce that, for $A, B, C, D \in \mathcal{H}_N$,

$$d_H(A \cup B, C \cup D) \leqslant \max\{d_H(A, C), d_H(B, D)\}.$$

We have

$$\begin{aligned} \rho(A \cup B, C) &= \max\{d(x, C) : x \in A \cup B\} \\ &= \max\{\max_{a \in A} d(a, C), \max_{b \in B} d(b, C)\} \\ &= \max\{\rho(A, C), \rho(B, C)\} \end{aligned}$$

and

$$\begin{aligned} \rho(A, B \cup C) &= \max_{a \in A} d(a, B \cup C) \\ &= \max_{a \in A}(\min\{d(a, B), d(a, C)\}) \\ &\leqslant \min\{\max_{a \in A} d(a, B), \max_{a \in A} d(a, C)\} \\ &= \min\{\rho(A, B), \rho(A, C)\}, \end{aligned}$$

so

$$\rho(A \cup B, C \cup D) = \max\{\rho(A, C \cup D), \rho(B, C \cup D)\} \leqslant \max\{\rho(A, C), \rho(B, D)\}.$$

Interchanging A with C and B with D yields

$$\rho(C \cup D, A \cup B) \leqslant \max\{\rho(C, A), \rho(D, B)\}.$$

Therefore

$$\begin{aligned} d_H(A \cup B, C \cup D) &= \max\{\rho(A \cup B, C \cup D), \rho(C \cup D, A \cup B)\} \\ &\leqslant \max\{\max\{\rho(A, C), \rho(B, D)\}, \max\{\rho(C, A), \rho(D, B)\}\} \\ &= \max\{\rho(A, C), \rho(B, D), \rho(C, A), \rho(D, B)\} \\ &= \max\{d_H(A, C), d_H(B, D)\}. \end{aligned}$$

PMA443 Fractals 2009–10
Question Sheet 5

To be handed in on Thursday 18 March.

1. Given a constant $0 < s < 1$, let $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$ be the IFS on $\mathbb{R}^2$ given by

$$w_1(x, y) = (sx, 1 - s + sy), \qquad w_2(x, y) = (1 - s + sx, 1 - s + sy),$$

$$w_3(x, y) = (1 - s + sx, sy), \qquad w_4(x, y) = (sx, sy).$$

Sketch the attractor of $\mathcal{W}$ in the case $s = 1/3$ and describe its relation to the Cantor ternary set C. What happens to the attractor for $s \geqslant 1/2$?

2. Let $\mathcal{W} = \{w_1, w_2, w_3\}$ be the IFS on $\mathbb{R}^2$ given by

$$\begin{aligned} w_1(x, y) &= \left(\frac{x}{2}, \frac{y+1}{2}\right) \\ w_2(x, y) &= \left(\frac{x+1}{2}, \frac{y+1}{2}\right) \\ w_3(x, y) &= \left(\frac{x}{2}, \frac{y}{2}\right). \end{aligned}$$

Let A be the attractor of $\mathcal{W}$. Let K_1 be the perimeter of the unit square:

$$K_1 = \{(x, y) \in [0, 1]^2 : \max\{x, y\} = 1 \text{ or } \min\{x, y\} = 0\}.$$

Let K_2 be the triangle

$$K_2 = \{(x, y) \in [0, 1]^2 : x + y \leqslant 1\}.$$

(These sets and the iterates $W^n(K_1)$ and $W^n(K_2)$ are illustrated in one of the recent handouts.)

What are the Lipschitz constants of the mappings w_i? What is the Lipschitz constant of the mapping $W : \mathcal{H}_2 \to \mathcal{H}_2$ associated with $\mathcal{W}$? Barnsley's Collage Theorem, in one form, tells us that

$$d_H(K_i, A) \leqslant \frac{d_H(K_i, W(K_i))}{1 - \operatorname{Lip} W}.$$

Evaluate $d_H(K_i, W(K_i))$ and $d_H(K_i, A)$ for $i = 1, 2$ and so check that Barnsley's Collage Theorem holds in these two cases.

Give an example of a non-empty compact set K_3 (a subset of $\mathbb{R}^2$, but not necessarily of $[0, 1]^2$), with $K_3 \neq A$, for which the Collage Theorem's conclusion is best possible.

(Proofs of your evaluations are not required.)

PMA443 Fractals 2009–10
Question Sheet 5: Solutions

1. *Given a constant $0 < s < 1$, let $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$ be the IFS on $\mathbb{R}^2$ given by*

$$w_1(x, y) = (sx, 1 - s + sy), \qquad w_2(x, y) = (1 - s + sx, 1 - s + sy),$$

$$w_3(x, y) = (1 - s + sx, sy), \qquad w_4(x, y) = (sx, sy).$$

Sketch the attractor of $\mathcal{W}$ in the case $s = 1/3$ and describe its relation to the Cantor ternary set C. What happens to the attractor for $s \geqslant 1/2$?

For $s = 1/3$ the attractor is $C \times C$. Once we have guessed this, proving it is simply a matter of showing that $C \times C$ is self-similar for $\mathcal{W}$, because $C \times C$ is a non-empty compact set and the attractor is unique. The fact that $C \times C$ is self-similar for $\mathcal{W}$ is immediate from the fact that the mappings $x \mapsto sx$ and $x \mapsto 1 - s + sx$ form an IFS with attractor C.

Likewise, for $s \geqslant 1/2$, we only need find a compact non-empty set which is self-similar for $\mathcal{W}$ and the (filled-in) unit square $S = [0, 1] \times [0, 1]$ is such a set; it is therefore the attractor. For $s < 1/2$ the attractor looks like the $s = 1/3$ picture, as can be seen by starting with the unit square and forming the sequence of iterates $W^n(S)$. We see that $S \supseteq W(S) \supseteq W^2(S) \supseteq \ldots$; the attractor is (in this case) the intersection.

2. *Let $\mathcal{W} = \{w_1, w_2, w_3\}$ be the IFS on $\mathbb{R}^2$ given by $w_1(x, y) = \left(\frac{x}{2}, \frac{y+1}{2}\right)$, $w_2(x, y) = \left(\frac{x+1}{2}, \frac{y+1}{2}\right)$ and $w_3(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$. Let A be the attractor of $\mathcal{W}$. Let K_1 be the perimeter of the unit square:*

$$K_1 = \{(x, y) \in [0, 1]^2 : \max\{x, y\} = 1 \text{ or } \min\{x, y\} = 0\}.$$

Let K_2 be the triangle

$$K_2 = \{(x, y) \in [0, 1]^2 : x + y \leqslant 1\}.$$

(These sets and the iterates $W^n(K_1)$ and $W^n(K_2)$ are illustrated in one of the recent handouts.)

What are the Lipschitz constants of the mappings w_i? What is the Lipschitz constant of the mapping $W : \mathcal{H}_2 \to \mathcal{H}_2$ associated with $\mathcal{W}$? Barnsley's Collage Theorem, in one form, tells us that

$$d_H(K_i, A) \leqslant \frac{d_H(K_i, W(K_i))}{1 - \operatorname{Lip} W}.$$

Evaluate $d_H(K_i, W(K_i))$ and $d_H(K_i, A)$ for $i = 1, 2$ and so check that Barnsley's Collage Theorem holds in these two cases.

Give an example of a non-empty compact set K_3 (a subset of $\mathbb{R}^2$, but not necessarily of $[0, 1]^2$), with $K_3 \neq A$, for which the Collage Theorem's conclusion is best possible.

The w_i are similitudes with $\operatorname{Lip} w_i = 1/2$ for each i. Hence $\operatorname{Lip} W \leqslant 1/2$. To see that $\operatorname{Lip} W = 1/2$, we observe that if A, B are the singleton sets $A = \{(0,0)\}, B = \{(0,1)\}$, then

$$\begin{aligned} W(A) &= \{(0,0),(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2})\} \\ W(B) &= \{(0,1),(0,\frac{1}{2}),(\frac{1}{2},1)\}, \end{aligned}$$

and so

$$d_H(W(A),W(B)) = \frac{1}{2} = \frac{1}{2} d_H(A,B).$$

By inspection, we get the same results for $i = 1$ and $i = 2$:

$$d_H(K_i, W(K_i)) = \frac{1}{2}, \qquad d_H(K_i, A) = \frac{1}{\sqrt{2}}.$$

Comparing this with Barnsley's Collage Theorem:

$$d_H(K_i, A) \leqslant \frac{d_H(K_i, W(K_i))}{1-s},$$

where $s = 1/2$, we see that the Collage Theorem estimate is satisfied with a factor of $1/\sqrt{2}$ to spare.

As an example of a suitable K_3, take any singleton set, for example $\{(1,0)\}$. Then

$$W(K_3) = \{(\frac{1}{2},\frac{1}{2}),(1,\frac{1}{2}),(\frac{1}{2},0)\},$$

so

$$d_H(K_3, W(K_3)) = \frac{1}{\sqrt{2}}, \qquad d_H(K_3, A) = \sqrt{2},$$

and so

$$d_H(K_3, A) = \frac{d_H(K_3, W(K_3))}{1-s}.$$

PMA443 Fractals 2009–10
Question Sheet 6

Not to be handed in. Solutions will be posted on Thursday 15 April.

1. Prove from the definition of Kolmogorov dimension that every nonempty finite set has Kolmogorov dimension zero.

2. Find a sequence of finite subsets $F_n \subseteq \mathbb{R}$ such that $F_n \to C$ as $n \to \infty$ in $\mathcal{H}_1$, where C is the Cantor ternary set. Deduce that it is not generally true that $A_n \to A$ implies $\mathrm{Kdim} A_n \to \mathrm{Kdim} A$.

3. For compact subsets A, B of $\mathbb{R}^N$, let $N(A, \varepsilon), N(B, \varepsilon)$ be the minimum number of ε-balls, centred on any points of $\mathbb{R}^N$, needed to cover A, B, respectively. Show that if $d_H(A, B) < \varepsilon$ (the Hausdorff metric), then $N(A, 2\varepsilon) \leqslant N(B, \varepsilon)$ and $N(B, 2\varepsilon) \leqslant N(A, \varepsilon)$.

   "It follows that if $A_n \to A$ in $\mathcal{H}_N$, then

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Kdim} A_n &= \lim_{n\to\infty,\varepsilon\to 0} \frac{\log N(A_n, \varepsilon)}{\log(1/\varepsilon)} \\ &\geqslant \lim_{\varepsilon\to 0} \frac{\log N(A, 2\varepsilon)}{\log(1/2\varepsilon) + \log 2} \\ &= \mathrm{Kdim} A\end{aligned}$$

   and

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Kdim} A_n &= \lim_{n\to\infty,\varepsilon\to 0} \frac{\log N(A_n, 2\varepsilon)}{\log(1/2\varepsilon)} \\ &\leqslant \lim_{\varepsilon\to 0} \frac{\log N(A, \varepsilon)}{\log(1/\varepsilon) - \log 2} \\ &= \mathrm{Kdim} A.\end{aligned}$$

   Thus $\mathrm{Kdim} A_n \to \mathrm{Kdim} A$."

   Explain where the above argument fails. (It certainly does fail, because you have given a counterexample in Question 2.)

PMA443 Fractals 2009–10
Question Sheet 6: Solutions

1. *Prove from the definition of Kolmogorov dimension that every nonempty finite set has Kolmogorov dimension zero.*

   Let $K = \{x_1, \ldots, x_n\}$ be a nonempty finite set. Then for all $\varepsilon > 0$,
$$K \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon).$$
   Therefore $N(\varepsilon) \leqslant n$, so
$$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \leqslant \frac{\log n}{\log(1/\varepsilon)} \to 0$$
   as $\varepsilon \to 0$. By the Sandwich Rule,
$$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \to 0,$$
   i.e. $\mathrm{Kdim} K$ exists, equal to zero.

2. *Find a sequence of finite subsets $F_n \subseteq \mathbb{R}$ such that $F_n \to C$ as $n \to \infty$ in $\mathcal{H}_1$, where C is the Cantor ternary set. Deduce that it is not generally true that $A_n \to A$ implies $\mathrm{Kdim} A_n \to \mathrm{Kdim} A$.*

   Let F_n be the set of end-points of intervals of the sets C_n defined in the construction of C. Then $d_H(F_n, C) = 3^{-(n+1)} \to 0$ as $n \to \infty$. Finite sets have Kolmogorov dimension zero, so
$$\mathrm{Kdim} F_n = 0 \not\to \frac{\log 2}{\log 3} = \mathrm{Kdim} C.$$

3. *For compact subsets A, B of $\mathbb{R}^N$, let $N(A, \varepsilon), N(B, \varepsilon)$ be the minimum number of ε-balls, centred on any points of $\mathbb{R}^N$, needed to cover A, B, respectively. Show that if $d_H(A, B) < \varepsilon$ (the Hausdorff metric), then $N(A, 2\varepsilon) \leqslant N(B, \varepsilon)$ and $N(B, 2\varepsilon) \leqslant N(A, \varepsilon)$.*

   "It follows that if $A_n \to A$ in $\mathcal{H}_N$, then
$$\begin{aligned} \lim_{n\to\infty} \mathrm{Kdim} A_n &= \lim_{n\to\infty, \varepsilon\to 0} \frac{\log N(A_n, \varepsilon)}{\log(1/\varepsilon)} \\ &\geqslant \lim_{\varepsilon\to 0} \frac{\log N(A, 2\varepsilon)}{\log(1/2\varepsilon) + \log 2} \\ &= \mathrm{Kdim} A \end{aligned}$$
   and
$$\begin{aligned} \lim_{n\to\infty} \mathrm{Kdim} A_n &= \lim_{n\to\infty, \varepsilon\to 0} \frac{\log N(A_n, 2\varepsilon)}{\log(1/2\varepsilon)} \\ &\leqslant \lim_{\varepsilon\to 0} \frac{\log N(A, \varepsilon)}{\log(1/\varepsilon) - \log 2} \\ &= \mathrm{Kdim} A. \end{aligned}$$

Thus $\text{Kdim} A_n \to \text{Kdim} A$.*"*

Explain where the above argument fails.

Since $d_H(A,B) < \varepsilon$, it follows that for every $a \in A$ there exists $b \in B$ with $d(a,b) < \varepsilon$. Then b is contained in one of the set of $N(B,\varepsilon)$ ε-balls with which we can cover B. Therefore a lies in one of the 2ε-balls with the same centres. This shows that $N(A, 2\varepsilon) \leqslant N(B,\varepsilon)$. The fact that $N(B, 2\varepsilon) \leqslant N(A,\varepsilon)$ follows simply by interchanging the rôles of A and B.

The problem with the type of argument suggested in the question is essentially one of interchanging the two limiting operations $\lim_{n\to\infty}$ and $\lim_{\varepsilon\to 0}$. To form

$$\lim_{n\to\infty} \text{Kdim} A_n,$$

one has to first take a $\lim_{\varepsilon\to 0}$ and then a $\lim_{n\to\infty}$. However, the argument needs first to make n sufficiently large to get $d_H(A_n, A) < \varepsilon$, say $n \geqslant n_0(\varepsilon)$, then to let $\varepsilon \to 0$. This amounts to doing $\lim_{n\to\infty}$ before $\lim_{\varepsilon\to 0}$.

PMA443 Fractals 2009–10
Question Sheet 7

Not to be handed in. Solutions to the first two questions will be posted on Thursday 22 April.

1. Show that, in the definition of Kolmogorov dimension of a subset K of a metric space X, it does not matter whether we require the centres of the ε-balls covering K to be in K (as we did) or allow the centres to be any points of X. [Hint: see Question Sheet 3, Question 3, about the definition of total boundedness, where there are also two equivalent definition — and use the Comparison Lemma.]

2. Let $f : X \to Y$ be a Lipschitz mapping between metric spaces and K is a non-empty compact subset of X which has Kolmogorov dimension. Show that **if** $f(K)$ (which is necessarily a non-empty compact subset of Y) also has Kolmogorov dimension, then $\mathrm{Kdim} f(K) \leqslant \mathrm{Kdim} K$. [Hint: recall Question Sheet 3, Question 5:

   > Show that a non-constant mapping $f : X \to Y$ between metric spaces X and Y is Lipschitz with Lipschitz constant less than or equal to λ if and only if
   > $$f(B(x, \varepsilon)) \subseteq B(f(x), \lambda\varepsilon)$$
   > for all $x \in X$ and $\varepsilon > 0$.

3. Using Google or otherwise find some examples of uses of 'fractal dimension' outside pure mathematics. We shall compare notes on 22 April.

PMA443 Fractals 2009–10
Question Sheet 7: Solutions

1. *Show that, in the definition of Kolmogorov dimension of a subset K of a metric space X, it does not matter whether we require the centres of the ε-balls covering K to be in K (as we did) or allow the centres to be any points of X. [Hint: see Question Sheet 3, Question 3, about the definition of total boundedness, where there are also two equivalent definition — and use the Comparison Lemma.]*

   Here is the solution to the total boundedness problem:

   > Suppose K satisfies the definition of total boundedness without the restriction that the centres of the ε-balls lie in K. Then, given $\varepsilon > 0$, there is a finite set $\{y_1, y_2, \ldots, y_n\} \subseteq X$ such that
   >
   > $$K \subseteq B(y_1, \varepsilon/2) \cup B(y_2, \varepsilon/2) \cup \ldots \cup B(y_n, \varepsilon/2).$$
   >
   > By removing any redundant balls from this covering, if necessary, we may assume that for each i there is a point $x_i \in K \cap B(y_i, \varepsilon/2)$. By the triangle inequality, $B(y_i, \varepsilon/2) \subseteq B(x_i, \varepsilon)$, so
   >
   > $$K \subseteq B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots \cup B(x_n, \varepsilon),$$
   >
   > as desired.

   For a compact set K, let $N(\varepsilon)$ be the minimum number of ε-balls centred on points of K needed to cover K and $N'(\varepsilon)$ be the minimum number of ε-balls centred on points of X needed to cover K. Clearly

   $$N'(\varepsilon) \leqslant N(\varepsilon), \tag{1}$$

   since any covering by balls centred on points of K is a covering by balls centred on points of X.

   The converse is trickier, but the argument cited above showed that for every covering by $\varepsilon/2$-balls there is a covering by the same number of ε-balls centred on points of K. Thus

   $$N(\varepsilon) \leqslant N'(\varepsilon/2). \tag{2}$$

   Applying the Comparison Lemma to (1) and (2) shows that

   $$\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

   exists if and only if

   $$\lim_{\varepsilon \to 0} \frac{\log N'(\varepsilon)}{\log(1/\varepsilon)}$$

   and they are equal. This is the desired result.

2. *Let $f : X \to Y$ be a Lipschitz mapping between metric spaces and K is a non-empty compact subset of X which has Kolmogorov dimension. Show that* **if** *$f(K)$ (which is necessarily a non-empty compact subset of Y) also has Kolmogorov dimension, then* $\mathrm{Kdim} f(K) \leqslant \mathrm{Kdim} K$.

   Let us write $L = f(K)$ and denote by $N_K(\varepsilon)$, $N_L(\varepsilon)$ the minimum number of ε-balls needed to cover K, L, respectively.

   If f is a constant mapping, then $f(K)$ is a single point and so has dimension 0 and the result holds. Otherwise, suppose

$$d(f(x_1), f(x_2)) \leqslant \lambda d(x_1, x_2) \qquad (x_1, x_2 \in X).$$

   Given $\varepsilon > 0$,

$$f(B(x, \varepsilon)) \subseteq B(f(x), \lambda\varepsilon)) \qquad (x \in X).$$

   Therefore, if

$$K \subseteq \bigcup_{i=1}^{N(\varepsilon)} B(x_i, \varepsilon)$$

   then

$$L = f(K) \subseteq \bigcup_{i=1}^{N(\varepsilon)} f(B(x_i, \varepsilon)) \subseteq \bigcup_{i=1}^{N(\varepsilon)} B(f(x_i), \lambda\varepsilon)).$$

   Thus $N_L(\lambda\varepsilon) \leqslant N_K(\varepsilon)$.

   We do not necessarily have an inequality in the reverse direction. However, we do know that both K and L have Kolmogorov dimension, so

$$\mathrm{Kdim} K = \lim_{\varepsilon \to 0} \frac{\log N_K(\varepsilon)}{\log(1/\varepsilon)} \quad \text{and} \quad \mathrm{Kdim} L = \lim_{\varepsilon \to 0} \frac{\log N_L(\lambda\varepsilon)}{\log(1/\lambda\varepsilon)}$$

   both exist. Since $N_L(\lambda\varepsilon) \leqslant N_K(\varepsilon)$, it follows that

$$\frac{\log N_L(\lambda\varepsilon)}{\log(1/\lambda\varepsilon)} \frac{\log(1/\lambda\varepsilon)}{\log(1/\varepsilon)} \leqslant \frac{\log N_K(\varepsilon)}{\log(1/\varepsilon)}$$

   when $\varepsilon < 1$. Now

$$\frac{\log(1/\lambda\varepsilon)}{\log(1/\varepsilon)} = \frac{\log(1/\varepsilon) - \log\lambda}{\log(1/\varepsilon)} = 1 - \frac{\log\lambda}{\log(1/\varepsilon)} \to 1$$

   as $\varepsilon \to 0$, and hence, taking limits, $\mathrm{Kdim} L \leqslant \mathrm{Kdim} K$.

PMA443 Fractals 2009–10
Question Sheet 8

Not to be handed in. Solutions will be posted on Thursday 29 April.

1. Using the fact that $\mathrm{Kdim}[0,1]^n = n$ and $\mathrm{Kdim}C = (\log 2)/\log(3)$, (where C is the Cantor set), and the standard theorems about the dimension of unions, Lipschitz images, *et cetera*, find the Kolmogorov dimension of each of the following sets:
   (a) the interval $[1, 5]$;
   (b) the set $\{0\} \cup [1, 5]$;
   (c) the boundary B of the unit square in $\mathbb{R}^2$;
   (d) the closed unit disc in $\mathbb{R}^2$;
   (e) the set $\{(x, y) \in [0, 1]^2 : x \in C \text{ or } y \in C\}$.
   (f) the interval $[0, \infty)$ (trick question).

2. For a real number $s > 1$, consider the compact set

$$X = \{\frac{1}{n^s} : n = 1, 2, 3, \ldots\} \cup \{0\} \subseteq \mathbb{R}$$

   as a subset of $\mathbb{R}$. Prove that $\mathrm{Kdim}X = 1/(s + 1)$. (Hint: to estimate

$$\left|\frac{1}{p^s} - \frac{1}{q^s}\right|,$$

   use the Mean Value Theorem applied to the function $f(x) = x^{-s}$ in the interval $[p, q]$.)

3. For $n = 1, 2, 3, \ldots$, let S_n be the circle in $\mathbb{R}^2$ centre 0, radius $1/n$, i.e.

$$S_n = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 1/n\}.$$

   Show that the set

$$S = \bigcup_{n=1}^{\infty} S_n \cup \{0\}$$

   has Kolmogorov dimension 1. (Hint: you may find the inequality

$$\sum_{n=1}^{m} \frac{1}{n} \leqslant 1 + \log m$$

   useful.)

4. Let $A = \{2^{-n} : n = 0, 1, 2, \ldots\} \cup \{0\} \subseteq \mathbb{R}$ with the usual metric. Prove that $\mathrm{Kdim}A = 0$.

PMA443 Fractals 2009–10
Question Sheet 8: Solutions

1. *Using the fact that* $\mathrm{Kdim}[0,1]^n = n$ *and the standard theorems about the dimension of unions, Lipschitz images, et cetera, find the Kolmogorov dimension of each of the following sets: (a) the interval* $[1,5]$*; (b) the set* $\{0\} \cup [1,5]$*; (c) the boundary* B *of the unit square in* $\mathbb{R}^2$*; (d) the closed unit disc in* $\mathbb{R}^2$*; (e) the set* $\{(x,y) \in [0,1]^2 : x \in C \text{ or } y \in C\}$*. (f) the interval* $[0,\infty)$ *(trick question).*

   In what follows, saying '$\mathrm{Kdim}K = \ldots$' should be taken to mean 'the set K **has Kolmogorov dimension** and $\mathrm{Kdim}K = \ldots$'.

   (a) The interval $[1,5]$ is the image of $[0,1]$ under the biLipschitz map $f(x) = 4x+1$. Therefore $\mathrm{Kdim}[1,5] = \mathrm{Kdim}[0,1] = 1$.

   (b) The Kolmogorov dimension of a singleton is zero, so
   $$\mathrm{Kdim}(\{0\} \cup [1,5]) = \max\{\mathrm{Kdim}\{0\}, \mathrm{Kdim}[1,5]\} = \max\{0,1\} = 1.$$

   (c) The set B is the union of four compact sets, the sides of the square, each of which is biLipschitz with the interval $[0,1]$ and so has Kolmogorov dimension 1. Therefore B has Kolmogorov dimension 1.

   (d) Let D denote the closed unit disc in $\mathbb{R}^2$; let S_1, S_2 denote the squares $[-1/\sqrt{2}, +1/\sqrt{2}]^2$ and $[-1,1]^2$, respectively. The biLipschitz maps
   $$f : [0,1]^2 \to S_1 : (x,y) \mapsto ((2x-1)/\sqrt{2}, (2y-1)/\sqrt{2}),$$
   $$f : [0,1]^2 \to S_2 : (x,y) \mapsto ((2x-1), (2y-1))$$
   give us
   $$\mathrm{Kdim}S_1 = \mathrm{Kdim}S_2 = 2.$$
   Now
   $$S_1 \subseteq D \subseteq S_2,$$
   so $\mathrm{Kdim}D = 2$.

   (e) We can write
   $$\{(x,y) \in [0,1]^2 : x \in C \text{ or } y \in C\} = (C \times [0,1]) \cup ([0,1] \times C).$$
   Now
   $$\mathrm{Kdim}(C \times [0,1]) = \mathrm{Kdim}C + \mathrm{Kdim}[0,1] = (\log 2)/(\log 3) + 1 = (\log 6)/(\log 3).$$
   Similarly, $\mathrm{Kdim}([0,1] \times C) = (\log 6)/(\log 3)$. Therefore
   $$\begin{aligned} \mathrm{Kdim}\{(x,y) \in [0,1]^2 : x \in C \text{ or } y \in C\} &= \mathrm{Kdim}\left((C \times [0,1]) \cup ([0,1] \times C)\right) \\ &= \max\left\{\mathrm{Kdim}(C \times [0,1]), \mathrm{Kdim}([0,1] \times C)\right\} \\ &= (\log 6)/(\log 3). \end{aligned}$$

(f) The interval $[0, \infty)$ is not compact, and Kolmogorov dimension is defined only for compact sets.

2. *For a real number $s > 1$, consider the compact set*

$$X = \{\frac{1}{n^s} : n = 1, 2, 3, \ldots\} \cup \{0\} \subseteq \mathbb{R}$$

as a subset of $\mathbb{R}$. *Prove that* $\mathrm{Kdim} X = 1/(s+1)$. *(Hint: to estimate*

$$\left| \frac{1}{p^s} - \frac{1}{q^s} \right|,$$

use the Mean Value Theorem applied to the function $f(x) = x^{-s}$ in the interval $[p, q]$.)

Given $\varepsilon \in (0, 1/2)$, if $n = n(\varepsilon)$ is the positive integer such that

$$\frac{1}{2(n+1)^{s+1}} < \varepsilon \leqslant \frac{1}{2n^{s+1}},$$

then X can be covered by ε-balls centred on the $2n+1$ points

$$\frac{k + \frac{1}{2}}{(n+1)^{s+1}} \qquad (0 \leqslant k < n)$$

and

$$\frac{1}{\ell^s} \qquad (1 \leqslant \ell \leqslant n).$$

Thus $N(\varepsilon) \leqslant 2n + 1$.

If $p \neq q$ are positive integers and $p, q \leqslant n$, then

$$d\left(\frac{1}{p^s}, \frac{1}{q^s}\right) \geqslant \frac{1}{(n-1)^s} - \frac{1}{n^s} > \frac{s}{n^{s+1}} > \frac{1}{n^{s+1}} \geqslant 2\varepsilon.$$

(The second inequality comes from applying the Mean Value Theorem to the function $f(x) = -x^{-s}$ in the interval $[n-1, n]$.) Hence, at least n ε-balls are required to cover X: so $N(\varepsilon) \geqslant n$. Therefore

$$\begin{aligned}
\frac{\log n}{\log 2 + (s+1)\log(n+1)} &= \frac{\log n}{\log\left(2(n+1)^{s+1}\right)} \\
&\leqslant \frac{\log(N(\varepsilon)}{\log(1/\varepsilon)} \\
&\leqslant \frac{\log(2n+1)}{\log(2n^{s+1})} \\
&= \frac{\log 2 + \log\left(n + \frac{1}{2}\right)}{\log 2 + (s+1)\log n}.
\end{aligned}$$

As $\varepsilon \to 0$, we have $n \to \infty$ and the first and last terms both tend to $1/(s+1)$, so the Kolmogorov dimension of K exists and is equal to $1/(s+1)$.

3. *For* $n = 1, 2, 3, \ldots$, *let* S_n *be the circle in* $\mathbb{R}^2$ *centre 0, radius* $1/n$, *i.e.*

$$S_n = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 1/n\}.$$

Show that the set

$$S = \bigcup_{n=1}^{\infty} S_n \cup \{0\}$$

has Kolmogorov dimension 1. (Hint: you may find the inequality

$$\sum_{n=1}^{m} \frac{1}{n} \leqslant 1 + \log m$$

useful.)

Given $\varepsilon > 0$, let $m = [1/\varepsilon]$. Then we need at least m balls of radius ε to cover the circle $\{(x, y) : x^2 + y^2 = 1\} \subseteq S$. This is a gross underestimate: we would need this number to cover a diameter of the circle. Thus $N(\varepsilon) \geqslant m$.

Conversely, we can cover the circle $\{(x, y) : x^2 + y^2 = 1/n^2\}$ by $[\pi/n\varepsilon] + 1$ ε-balls. Then, since $\varepsilon \geqslant 1/m$,

$$B(0, \varepsilon) \supseteq \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \in \{1/n^2 : n > m\} \cup \{0\}\}.$$

Thus we can cover S with

$$1 + \sum_{n=1}^{m} \left(\left[\frac{\pi}{n\varepsilon} \right] + 1 \right)$$

ε-balls. Thus

$$\begin{aligned} m &\leqslant N(\varepsilon) \\ &\leqslant 1 + \sum_{n=1}^{m} \left(\left[\frac{\pi}{n\varepsilon} \right] + 1 \right) \\ &\leqslant m + 1 + \sum_{n=1}^{m} \frac{\pi}{n\varepsilon} \\ &\leqslant m + 1 + \frac{\pi}{\varepsilon}(1 + \log m). \end{aligned}$$

Therefore

$$\begin{aligned} 1 &= \frac{\log m}{\log m} \\ &\leqslant \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \\ &\leqslant \frac{\log(m + 1 + \pi(m + 1)(1 + \log m))}{\log m} \\ &\leqslant \frac{\log(4m \log m)}{\log m}, \end{aligned}$$

for sufficiently large m. As $\varepsilon \to 0$ and so $m \to \infty$, the last term tends to 1. Thus Kdim$S$ exists and equals 1.

4. *Let* $A = \{2^{-n} : n = 0, 1, 2, \ldots\} \cup \{0\} \subseteq \mathbb{R}$ *with the usual metric. Prove that* $\mathrm{Kdim} A = 0$.

For $1 > \varepsilon > 0$, let $n \geqslant 1$ be such that $2^{-n} < 2\varepsilon < 2^{-n+1}$. Let B_0 be an ε-ball containing $[0, 2^{-n}]$ and for $1 \leqslant i \leqslant n$ let $B_i = B(2^{-i+1}, \varepsilon)$. Then

$$A \subseteq \bigcup_{i=0}^{n} B_i.$$

Thus

$$\begin{aligned} N(\varepsilon) &\leqslant n + 1 \\ &\leqslant \log_2(1/2\varepsilon) + 2 \\ &\leqslant \log_2(1/\varepsilon) + 2 \\ &\leqslant 2 \ln 1/(\varepsilon), \end{aligned}$$

for all sufficiently small ε, so

$$\begin{aligned} 0 &\leqslant \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \\ &\leqslant \frac{\ln 2 + \ln\ln(1/\varepsilon)}{\ln(1/\varepsilon)} \\ &\to 0 \end{aligned}$$

as $\varepsilon \to 0$. So $\mathrm{Kdim} A$ exists, equal to 0.

PMA443 Fractals 2009–10
Question Sheet 9

Not to be handed in. Solutions will be posted on Thursday 6 May.

1. Show that if $A \subseteq B$ are subsets of a metric space X, then $\mathrm{Hdim} A \leqslant \mathrm{Hdim} B$.

2. Show that the unit interval $[0, 1]$ has Hausdorff dimension 1: you can split this into two parts.

   (a) Show that, for $d = 1$,

   $$[0,1] \subseteq \bigcup_{i \in I} B(a_i; \delta_i) \implies \sum_{i \in I} \delta_i^d \geqslant \frac{1}{2},$$

   by using the compactness of $[0, 1]$ to restrict attention to the case I finite.

   (b) Show that, for all $d > 1$ and $\varepsilon > 0$, there exists a cover

   $$[0,1] \subseteq \bigcup_{i \in I} B(a_i; \delta_i)$$

   with $\sum_{i \in I} \delta_i^d < \varepsilon$.

PMA443 Fractals 2009–10
Question Sheet 9: Solutions

1. *Show that if $A \subseteq B$ are subsets of a metric space X, then* $\mathrm{Hdim}A \leqslant \mathrm{Hdim}B$.

   If we have a covering
   $$\bigcup_{i=1}^{\infty} B(x_i, \delta_i)$$
   of B with
   $$\sum_{i=1}^{\infty} \delta_i^d < \varepsilon,$$
   then this is also a covering of A. It follows that if B is d-null, then so is A. Thus
   $$\{d : B \text{ is } d\text{-null}\} \subseteq \{d : A \text{ is } d\text{-null}\},$$
   so
   $$\inf\{d : B \text{ is } d\text{-null}\} \geqslant \inf\{d : A \text{ is } d\text{-null}\},$$
   i.e. $\mathrm{Hdim}B \geqslant \mathrm{Hdim}A$.

2. *Show that the unit interval* $[0, 1]$ *has Hausdorff dimension 1: you can split this into two parts.*

   (a) Show that, for $d = 1$,
   $$[0, 1] \subseteq \bigcup_{i \in I} B(a_i; \delta_i) \implies \sum_{i \in I} \delta_i^d \geqslant \frac{1}{2},$$
   by using the compactness of $[0, 1]$ *to restrict attention to the case I finite.*

   (b) Show that, for all $d > 1$ and $\varepsilon > 0$, there exists a cover
   $$[0, 1] \subseteq \bigcup_{i \in I} B(a_i; \delta_i)$$
   with $\sum_{i \in I} \delta_i^d < \varepsilon$.

   As suggested, we split the proof into two parts.

   (a) We show that, for $d = 1$,
   $$[0, 1] \subseteq \bigcup_{i \in I} B(a_i; \delta_i) \implies \sum_{i \in I} \delta_i^d \geqslant \frac{1}{2}.$$
   Since $[0, 1]$ is compact,
   $$[0, 1] \subseteq \bigcup_{i \in I} B(a_i; \delta_i) \implies [0, 1] \subseteq \bigcup_{i \in J} B(a_i; \delta_i)$$

for some finite $J \subseteq I$. By removing superfluous elements from J and renumbering, if necessary, we may assume that

$$a_1 - \delta_1 < 0 < a_2 - \delta_2 < a_1 + \delta_1 < a_3 - \delta_3 < a_2 + \delta_2 < \ldots < 1 < a_n + \delta_n.$$

Hence

$$2 \sum_{i \in J} \delta_i > 1,$$

so

$$\sum_{i \in I} \delta_i > \frac{1}{2}.$$

It follows that $\text{Hdim}[0,1] \geqslant 1$.

(b) We show that, for all $d > 1$ and $\varepsilon > 0$, there exists a cover

$$[0,1] \subseteq \bigcup_{i \in I} B(a_i; \delta_i)$$

with $\sum_{i \in I} \delta_i^d < \varepsilon$. This proves that $\text{Hdim}[0,1] \leqslant 1$.
For $n \in \mathbb{Z}^+$, let

$$a_i = \frac{i}{n+1}, \; \delta_i = \frac{1}{2n} \quad (0 \leqslant i \leqslant n+1).$$

Then

$$[0,1] \subseteq \bigcup_{i \in I} B(a_i; \delta_i)$$

and

$$\sum_{i=0}^{n+1} \delta_i^d = \frac{n+1}{(2n)^d} \to 0$$

as $n \to \infty$, since $d > 1$. The result follows by taking $n = n(\varepsilon, d)$ sufficiently large.

PMA443 Fractals 2009–10
Question Sheet 10

Not to be handed in. Solutions will be posted on Thursday 6 May.

1. Let us define a "deleted-fifths set" $D \subseteq \mathbb{R}$ by

$$D = \bigcap_{n=0}^{\infty} D_n,$$

where

$$\begin{aligned} D_0 &= [0,1], \\ D_1 &= [0,\frac{1}{5}] \cup [\frac{2}{5},\frac{3}{5}] \cup [\frac{4}{5},1], \\ D_2 &= [0,\frac{1}{25}] \cup [\frac{2}{25},\frac{3}{25}] \cup [\frac{4}{25},\frac{1}{5}] \cup [\frac{2}{5},\frac{11}{25}] \cup [\frac{12}{25},\frac{13}{25}] \cup [\frac{14}{25},\frac{3}{5}] \cup [\frac{4}{5},\frac{21}{25}] \cup [\frac{22}{25},\frac{23}{25}] \cup [\frac{24}{25},1], \\ &\cdots, \end{aligned}$$

each set D_n being formed from the previous one by deleting two fifths of each interval. Express D in terms of the expansions of numbers to base 5. Find an IFS of which D is the attractor and calculate its similarity dimension. Prove that your IFS satisfies the conditions of Hutchinson's Theorem. Hence find Hdim$D$.

2. Let $w_1, w_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$w_1(x,y) = \left(\frac{x}{3}, \frac{y}{3}\right), \qquad w_2(x,y) = \left(\frac{x+2}{3}, \frac{y}{3}\right), \qquad ((x,y) \in \mathbb{R}^2).$$

Find the similarity dimension of the IFS $\mathcal{W} = (w_1, w_2)$ and show that it satisfies the open set condition with

$$O = \{(s,t) : 0 < s,t < 1\}.$$

Use Hutchinson's Theorem to find the Hausdorff dimension of the attractor. What is the attractor?

3. Show that the Menger sponge is the attractor of an IFS satisfying the open set condition and hence calculate its Hausdorff dimension.

PMA443 Fractals 2009–10
Question Sheet 10: Solutions

1. *Let us define a "deleted-fifths set" $D \subseteq \mathbb{R}$ by*

$$D = \bigcap_{n=0}^{\infty} D_n,$$

where

$$\begin{aligned} D_0 &= [0,1], \\ D_1 &= [0,\tfrac{1}{5}] \cup [\tfrac{2}{5},\tfrac{3}{5}] \cup [\tfrac{4}{5},1], \\ D_2 &= [0,\tfrac{1}{25}] \cup [\tfrac{2}{25},\tfrac{3}{25}] \cup [\tfrac{4}{25},\tfrac{1}{5}] \cup [\tfrac{2}{5},\tfrac{11}{25}] \cup [\tfrac{12}{25},\tfrac{13}{25}] \cup [\tfrac{14}{25},\tfrac{3}{5}] \cup [\tfrac{4}{5},\tfrac{21}{25}] \cup [\tfrac{22}{25},\tfrac{23}{25}] \cup [\tfrac{24}{25},1], \\ \cdots &, \end{aligned}$$

each set D_n being formed from the previous one by deleting two fifths of each interval. Express D in terms of the expansions of numbers to base 5. Find an IFS of which D is the attractor and calculate its similarity dimension. Prove that your IFS satisfies the conditions of Hutchinson's Theorem. Hence find $\mathrm{Kdim} D$.

The set D is the set of all numbers having an expansion in base 5 of the form $0.d_1d_2d_3\ldots$ where $d_i \in \{0,2,4\}$ for all i.

Let

$$w_j(x) = \frac{x+j}{5} \qquad (j = 0,2,4).$$

Then w_j maps the number with base 5 expansion $0.d_1d_2d_3\ldots$ to $0.jd_1d_2d_3\ldots$. Hence

$$D = w_0(D) \cup w_2(D) \cup w_4(D).$$

Since D is compact and non-empty, it is the (necessarily unique) attractor of the IFS $\mathcal{W} = \{w_0, w_2, w_4\}$.

Each w_j is Lipschitz with $\mathrm{Lip}\, w_j = 1/5$, so the similarity dimension S (called D in the notes, but that notation is already in use here!) is given by

$$3\left(\frac{1}{5}\right)^S = 1,$$

$$\log 3 - S\log 5 = 0,$$

$$S = \frac{\log 3}{\log 5}.$$

The maps w_j of part (ii) are clearly similitudes:

$$|w_j(x) - w_j(y)| = \frac{1}{5}|x-y|,$$

with the same Lipschitz constant.

To check the open set condition, let $O = (0,1)$. Then

$$w_0(O) = \left(0, \frac{1}{5}\right), \quad w_2(O) = \left(\frac{2}{5}, \frac{3}{5}\right), \quad w_4(O) = \left(\frac{4}{5}, \frac{5}{5}\right).$$

It is then clear that

(a)

$$w_0(O) \cup w_2(O) \cup w_4(O) \subseteq O,$$

(b)

$$w_i(O) \cap w_j(O) = \emptyset \qquad (i \neq j).$$

Thus $\mathcal{W}$ satisfies the conditions of Hutchinson's Theorem and so the Hausdorff dimension is equal to the similarity dimension S.

2. *Let $w_1, w_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$w_1(x,y) = \left(\frac{x}{3}, \frac{y}{3}\right), \qquad w_2(x,y) = \left(\frac{x+2}{3}, \frac{y}{3}\right), \qquad ((x,y) \in \mathbb{R}^2).$$

Find the similarity dimension of the IFS $\mathcal{W} = (w_1, w_2)$ and show that it satisfies the open set condition with

$$O = \{(s,t) : 0 < s, t < 1\}.$$

Use Hutchinson's Theorem to find the Hausdorff dimension of the attractor. What is the attractor?

Both the w_i are similitudes with $\operatorname{Lip} w_i = 1/3$, so the similarity dimension is

$$D = \frac{\log M}{\log(1/s)} = \frac{\log 2}{\log 3}.$$

We have

$$\begin{aligned} w_1(O) &= \{(s,t) : 0 < s, t < 1/3\} \\ w_1(O) &= \{(s,t) : 2/3 < s < 1,\ 0 < t < 1/3\}. \end{aligned}$$

Thus $w_1(O)$ and $w_2(O)$ are disjoint and both contained in O.

Thus the conditions of Hutchinson's Theorem are satisfied and so the attractor has Hausdorff dimension $\log 2/\log 3$.

The attractor is a copy of the Cantor set. If C denotes the Cantor set, a subset of $[0,1]$, then the attractor is the set $A = \{(x,0) : x \in C\}$. To prove this, we need only observe that A is a compact subset of $\mathbb{R}^2$ which is self-similar with respect to the given IFS.

3. *Show that the Menger sponge is the attractor of an IFS satisfying the open set condition and hence calculate its Hausdorff dimension.*

The Menger sponge is the union of 20 copies of itself, each one third the size of the original. For a picture of this, see
`http://www.3d-gfx.com/fractals/3d/bigmengersponge02.jpg`
The IFS $\mathcal{W} = \{w_1, \ldots, w_{20}\}$ is then almost obvious. All the w_i are similitudes with $\operatorname{Lip} w_i = 1/3$. Thus the w_i may be thought of as shrinking the sponge to one third size and positioning the shrunk portion suitably. There is an ambiguity about how to orientate the portion, due to the fact that the whole sponge has cubic symmetry; any orientation will do. To see that this IFS satisfies the open set condition, take the open set O in the definition to be the large open cube (which is best described as the interior of the convex hull of the sponge).

The similarity dimension of the IFS is $\log 20/\log 3$ and since the open set condition is satisfied, this is equal to the Hausdorff dimension.