PMA345 Functional Analysis June 2000 — Solutions

1. (i) [4 marks: bookwork definition] Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. By a norm on V is meant a function $\|\cdot\|$ mapping V into $\mathbb{R}_+$ for which, for $x, y \in V$ and $\lambda \in \mathbb{F}$:

(a) $\|x+y\| \leqslant \|x\| + \|y\|$;

(b) $\|\lambda x\| = |\lambda|\|x\|$;

(c) $\|x\| = 0$ if and only if $x = 0$.

We then say that $(V, \|\cdot\|)$ is a normed vector space.

1. (ii) [11 marks: bookwork; bookwork; unseen problem] (a) The sequence (λ_n) is a convergent sequence of numbers and so bounded. Choose $M \in \mathbb{R}$ such that $|\lambda_n| \leqslant M$ for all n. Then

$$\begin{aligned} 0 \leqslant \|\lambda_n x_n - \lambda x\| &= \|\lambda_n(x_n - x) + (\lambda_n - \lambda)x\| \\ &\leqslant |\lambda_n|\|x_n - x\| + |\lambda_n - \lambda|\|x\| \\ &\leqslant M\|x_n - x\| + \|x\||\lambda_n - \lambda| \\ &\to M.0 + \|x\|.0 = 0, \end{aligned}$$

as $n \to \infty$. So $\lambda_n x_n \to \lambda x$, by the sandwich rule.

(b) $\|x\| \leqslant \|x-y\| + \|y\|, \|y\| \leqslant \|y-x\| + \|x\| = \|x-y\| + \|x\|$.

So $-\|x-y\| \leqslant \|x\| - \|y\| \leqslant \|x-y\|$.

Therefore $|\|x\| - \|y\|| \leqslant \|x-y\|$.

Now let $x_n \to x$ in V. Then $\|x_n - x\| \to 0$. So, by the above, $|\|x_n\| - \|x\|| \to 0$. Therefore $\|x_n\| \to \|x\|$, as required.

(c) Write $\eta(x) = \|x\|$ for $x \in V$. Then we can write $U = \eta^{-1}((1,2))$. Since η is continuous and $(1,2)$ is open in $\mathbb{R}$ it follows that U is also open.

1. (iii) [10 marks: seen problem] Clearly $\|\cdot\|$ is a well-defined map from $V \times W$ to $\mathbb{R}$.

Let $(x,y), (u,v) \in V \times W$, and $\lambda \in \mathbb{F}$. Then

$$\|(x,y) + (u,v)\| = \|(x+u, y+v)\| = \max\{\|x+u\|, \|y+v\|\}.$$

Now $\|x+u\| \leqslant \|x\| + \|u\| \leqslant \max\{\|x\|, \|y\|\} + \max\{\|u\|, \|v\|\} = \|(x,y)\| + \|(u,v)\|$. Similarly $\|y+v\| \leqslant \|(x,y)\| + \|(u,v)\|$. So $\|(x,y) + (u,v)\| \leqslant \|(x,y)\| + \|(u,v)\|$.

Also $$\|\lambda(x,y)\| = \|(\lambda x, \lambda y)\| = \max\{\|\lambda x\|, \|\lambda y\|\} = \max\{|\lambda|\|x\|, |\lambda|\|y\|\} = |\lambda|\|(x,y)\|$$

Finally, if $\|(x,y)\| = \max\{\|x\|, \|y\|\} = 0$ then $x = 0$ and $y = 0$ so that $(x,y) = 0$.

So $V \times W$ is a NVS.

Now let $x_n \to x$ and $y_n \to y$ in V and W respectively. Then $\|(x_n, y_n) - (x,y)\| = \|(x_n - x, y_n - y)\|$. So $\|(x_n, y_n) - (x,y)\| = \max\{\|x_n - x\|, \|y_n - y\|\}$. But it is clear that this converges to 0 if and only if $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$, that is $x_n \to x$ and $y_n \to y$.

2. (i) [15 marks: all bookwork] (a) We say that T is bounded if there is a constant $K \in \mathbb{R}$ such that

$$\|Tx\| \leqslant K\|x\| \qquad \text{for all } x \in V.$$

We then define the norm $\|T\|$ of T to be the smallest value of K for which the above inequality is satisfied.

(b) Suppose that W is complete. Let (T_n) be a Cauchy sequence in $B[V, W]$. So given $\varepsilon > 0$, we can find N_1 such that

$$\|T_n - T_m\| < \varepsilon \qquad \text{if } n, m \geqslant N_1.$$

So therefore

$$\|T_n x - T_m x\| \leqslant \varepsilon\|x\| \qquad \text{if } x \in V, n, m \geqslant N_1. \qquad (*)$$

Fix $x \in V$. Then by $(*)$ the sequence $(T_n x)$ is Cauchy in W, and so convergent. Define $Tx = \lim T_n x$. As x varies, we get a map T from V to W.

Clearly T is linear since if $x, y \in V$ and $\lambda, \mu \in \mathbb{F}$ then

$$T(\lambda x + \mu y) = \lim T_n(\lambda x + \mu y) = \lim (\lambda T_n x + \mu T_n y) = \lambda Tx + \mu Ty.$$

Since (T_n) is Cauchy it is bounded, say $\|T_n\| \leqslant M$ for all n. Then for $x \in V$,

$$\|Tx\| = \lim \|T_n x\| \leqslant M\|x\|,$$

so that T is bounded, i.e. $T \in B[V, W]$.

Finally, in $(*)$, fix $n \geqslant N_1$ and let $m \to \infty$. So $T_n x - T_m x \to T_n x - Tx$, and so $\|T_n x - T_m x\| \to \|T_n x - Tx\|$. So $\|T_n x - Tx\| \leqslant \varepsilon\|x\|$ for all $x \in V$. Hence $\|T_n - T\| \leqslant \varepsilon$ if $n \geqslant N_1$. So $T_n \to T$. Hence $B[V, W]$ is complete.

2. (ii) [10 marks: unseen problem] Clearly $(T\mathbf{x})(n)$ is well defined for each $\mathbf{x}$ and each n. Also

$$|(T\mathbf{x})(n)| \leqslant \sum_{k=1}^{n} 2^{-k}|x(k)| \leqslant \sum_{k=1}^{n} 2^{-k}\|\mathbf{x}\| \leqslant 1\|\mathbf{x}\|.$$

So $T\mathbf{x}$ is a bounded sequence, and so is in ℓ^∞.

Also, from the same inequality, $\|T\mathbf{x}\| \leqslant 1\|\mathbf{x}\|$, so that T is bounded with norm at most 1 .

The sequence $\mathbf{u} = (1)$ is in ℓ^∞, and

$$T\mathbf{u}(n) = \sum_{k=1}^{n} 2^{-k} \to 1 \qquad \text{as } n \to \infty.$$

So $\|T\mathbf{u}\| = 1$. Since $\|\mathbf{u}\| = 1$ it is clear that $\|T\| \geqslant 1$. So $\|T\| = 1$.

3. (i) [8 marks: bookwork] A vector subspace W of V is called a hyperplane if $W \neq V$ and there is no subspace U of V with $W \subset U \subset V$.

Let f be a non-zero linear functional of V, and let W be the kernel of f. Then clearly W is a subspace of V. Since $f \neq 0$ it is clear that $W \neq V$. Let U be a subspace of V with W a proper subset of U. Choose $u \in U \setminus W$. Then $f(u) \neq 0$. So, for any $x \in V$, $f\left(x - \frac{f(x)}{f(u)}u\right) = f(x) - \frac{f(x)}{f(u)}f(u) = 0$. So that $x - \frac{f(x)}{f(u)}u \in W \subset U$. So $x = \left(x - \frac{f(x)}{f(u)}u\right) + \frac{f(x)}{f(u)}u \in U$. So $U = V$ as required.

3. (ii) [7 marks: problem] Let $f(\mathbf{x}) = 3x(1) - 2x(2)$ for $\mathbf{x} \in \ell^1$. Then $f(\mathbf{x}) = 0$ if and only if $3x(1) - 2x(2) = 0$, that is $\mathbf{x} \in H$. So H is the kernel of f. So H is a hyperplane. Also $|f(\mathbf{x})| = |3x(1) - 2x(2)| \leqslant 3(|x(1)| + |x(2)|) \leqslant 3\|\mathbf{x}\|$. So f is bounded, i.e. continuous. So we can deduce that H is closed by a lemma (2.6 (ii)).

3. (iii) [10 marks: seen] (a) Suppose that $\mathbf{y} \in \ell^1$. Define, for $x \in \ell^\infty$,

$$f_{\mathbf{y}}(\mathbf{x}) = \sum_{n=1}^{\infty} y(n)x(n).$$

Then

$$\begin{aligned} |f_{\mathbf{y}}(\mathbf{x})| &= \left|\sum_{n=1}^{\infty} y(n)x(n)\right| \\ &\leqslant \sum_{n=1}^{\infty} |y(n)||x(n)| \leqslant \sum_{n=1}^{\infty} |y(n)|\|\mathbf{x}\| \\ &= \|\mathbf{x}\| \sum_{n=1}^{\infty} |y(n)| = \|\mathbf{y}\|\|\mathbf{x}\|. \end{aligned}$$

Therefore $f_{\mathbf{y}}$ is bounded.

(b) Clearly $|g(\mathbf{x})| \leqslant \sup\{|x(n)| : n = 1, 2, 3, \ldots\} = \|\mathbf{x}\|$, so that g is continuous. By the Hahn-Banach theorem, we can extend g to a continuous linear functional f on ℓ^∞. Suppose that there is *any* sequence $\mathbf{y}$ in ℓ^1 such that

$$f(\mathbf{x}) = \sum_{n=1}^{\infty} y(n)x(n) \qquad (x \in \ell^\infty).$$

Then this is true for all $\mathbf{x} \in c$ when $f = g$. In particular, for $k = 1, 2, 3, \ldots$, let $\mathbf{x} = \mathbf{e_k}$ where $e_k(n)$ is equal to 1 if $n = k$ and to 0 if $n \neq k$. Then

$$0 = g(\mathbf{e_k}) = f(\mathbf{e_k}) = y(k).1 = y(k).$$

So $\mathbf{y} = 0$. So clearly $f = 0$ and hence $g = 0$. This is a contradiction, since if $\mathbf{x} = (1, 1, 1, \ldots)$ then $g(\mathbf{x}) = 1$.

4. (i) [8 marks: bookwork] (a) Uniform Boundedness theorem:
Let V and W be normed vector spaces with V complete. Let L be a subset of $B[V, W]$. The following are equivalent:

(a) L is a bounded set in $B[V, W]$;

(b) for each $x \in V$ the set $\{Tx : T \in L\}$ is bounded in W.

(b) If x is fixed then the sequence $(f_n(x))$, being convergent, is a bounded subset of $\mathbb{F}$. So, by the Uniform Boundedness theorem (with $W = \mathbb{F}$), the set of linear functionals $\{f_n : n = 1, 2, 3, \ldots\}$ is a bounded subset of $B[V, \mathbb{F}] = V'$, say $\|f_n\| \leqslant M$ for all n.So, if $x \in V$, then $|f_n(x)| \leqslant M\|x\|$, so

$$|f(x)| = \lim |f_n(x)| \leqslant \lim M\|x\| = M\|x\|,$$

so that f is bounded, and so is a continuous linear functional.

4. (ii) [11 marks: unseen] (a) For $n = 1, 2, 3, \ldots$, for each $\mathbf{x} \in \ell^1$,

$$|f_n(\mathbf{x})| \leqslant \sum_{k=1}^{n} |a(k)||x(k)| \leqslant \sum_{k=1}^{n} |x(k)| \max\{|a(k)| : 1 \leqslant k \leqslant n\} \leqslant \sum_{k=1}^{\infty} |x(k)| \max\{|a(k)| : 1 \leqslant k \leqslant n\},$$

so that f_n is bounded, and $\|f_n\| \leqslant \max\{|a(k)| : 1 \leqslant k \leqslant n\}$. But, for $1 \leqslant k \leqslant n$, if $a(k) \neq 0$, we can choose $\mathbf{x}_k \in \ell^1$ with $x_k(m) = 0$ if $m \neq k$, and $x_k(k) = a(k)/|a(k)|$.

Then $\|\mathbf{x}_k\| = 1$ (in ℓ^1) and $f_n(\mathbf{x}_k) = |a(k)|$.So therefore $\|f_n\| \geqslant |a(k)|$.
Hence $\|f_n\| = \max\{|a(k)| : 1 \leqslant k \leqslant n\}$.

(b) Now suppose that the series

$$\sum_{k=1}^{\infty} a(k)x(k)$$

is convergent, for all $\mathbf{x} = (x(1), x(2), \ldots x(n), \ldots) \in \ell^1$. Define a linear functional f on ℓ^1 by

$$f(\mathbf{x}) = \sum_{k=1}^{\infty} a(k)x(k) = \lim_{n\to\infty} \sum_{k=1}^{n} a(k)x(k) = \lim_{n\to\infty} f_n(\mathbf{x}).$$

By the Uniform Boundedness theorem, and the argument used in part (i), we see that there is M such that $\|f_n\| \leqslant M$ for all $n = 1, 2, 3, \ldots$, From our calculation above this means that $\max\{|a(k)| : 1 \leqslant k \leqslant n\} \leqslant M$, for all n. Hence the sequence $(a(n))$ is bounded.

4. (iii) [6 marks: unseen] Clearly, each f_n is well defined and linear. Also

$$|f_n(\mathbf{x})| \leqslant \sum_{k=1}^{n} 1/k|x(k)| \leqslant \sum_{k=1}^{n} 1/k\|\mathbf{x}\|,$$

so that f_n is bounded, with $\|f_n\| \leqslant \sum_{k=1}^{n} 1/k$. But if we take $\mathbf{x}$ to be the sequence $(1, 1, 1, \ldots, 0, 0, 0, \ldots)$, in $c_{00}(\mathbb{N})$ with n 1's , which has norm equal to 1, we see that $\|f_n\| = \sum_{k=1}^{n} 1/k$. Now, for any $\mathbf{x} = (x(1), x(2), x(3), \ldots x(N), 0, 0, \ldots)$ we have $f_n(\mathbf{x}) = \sum_{k=1}^{N} x(k)/k$ for all $n \geqslant N$, so that $\lim_{n\to\infty} f_n(\mathbf{x})$ exists.

Now $\sup\{\|f_n\| : n = 1, 2, 3, \ldots\} = \sum_{n=1}^{\infty} 1/k = \infty$. If $c_{00}(\mathbb{N})$ were complete then, from the argument used in the first part of the question, we should have $\sup\{\|f_n\| : n = 1, 2, 3, \ldots\}$ finite. This is not so, so $c_{00}(\mathbb{N})$ is not complete.

5. (i) [10 marks: unseen] Fix $n \in \mathbb{N}$. Then

$$0 \leqslant |x_k(n) - x(n)|^2 \leqslant \sum_{m=1}^{\infty} |x_k(m) - x(m)|^2 = \|\mathbf{x_k} - \mathbf{x}\|^2 \to 0.$$

So clearly $|x_k(n) - x(n)| \to 0$, that is $x_k(n) \to x(n)$.

Let $(\mathbf{x}_k)$ be as given. Then

$$\|\mathbf{x}_k\|^2 = 1^2 = 1,$$

so that $\|\mathbf{x}_k\| = 1$, that is $\mathbf{x}_k$ is in the unit ball of ℓ^2.

However, if the subsequence $(\mathbf{x}_{k_p})$ converges to $\mathbf{x}$ then for each n we have $x_{k_p}(n) = 0$ for $k_p > n$. So, since $k_p \to \infty$ it is clear that $x(n) = \lim_{p\to\infty} x_{p_k}(n) = 0$, so that $\mathbf{x} = \mathbf{0}$. But clearly $\|\mathbf{x}_{p_k} - 0\| = 1$. So the subsequence does not converge.

This shows that the unit ball is not compact, since we have a sequence which does not have a convergent subsequence.

5. (ii) [9 marks: bookwork] Let $\|\cdot\|_V$ be the given norm on N, and $\|\cdot\|_W$ the given norm on W. For $x \in V$ define

$$\|x\| = \|x\|_V + \|Tx\|_W.$$

Then for $x, y \in V,\ \lambda \in \mathbb{F}:$

$$\|x\| \geqslant \|x\|_V > 0 \quad \text{if } x \neq 0;$$

$$\|\lambda x\| = \|\lambda x\|_V + \|\lambda Tx\|_W = |\lambda|(\|x\|_V + \|Tx\|_W) = |\lambda|\|x\|;$$

$$\|x + y\| = \|x + y\|_V + \|Tx + Ty\|_W \leqslant \|x\| + \|y\| + \|Tx\|_W + \|Ty\|_W = \|x\| + \|y\|.$$

So $\|\cdot\|$ is another norm on V. So $\|\cdot\|$ is equivalent to $\|\cdot\|_V$. So there is K such that $\|x\| \leqslant K\|x\|_V$ for all $x \in V$. Therefore for all $x \in V$,

$$\|Tx\|_W \leqslant \|x\| \leqslant K\|x\|_V.$$

So T is bounded, i.e. continuous.

5. (iii) [6 marks: unseen] Define for $\mathbf{x} = (x(1), x(2), \ldots x(n)) \in \mathbb{C}^n$,

$$\|\mathbf{x}\| = \sum_{k=1}^{n} \frac{1}{k}|x(k)|.$$

Then it is clear that $\|\cdot\|$ is a norm on $\mathbb{C}^n$.

For if $\mathbf{x},\ \mathbf{y} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, then:

- $\|\mathbf{x}\| = 0$ if and only if $1/k\, x(k) = 0$ for all k, that is if and only if $\mathbf{x} = \mathbf{0}$;
- $\|\lambda\mathbf{x}\| = \sum_{k=1}^{n} |\lambda| \frac{1}{k}|x(k)| = |\lambda|\|\mathbf{x}\|$;
- $\|\mathbf{x} + \mathbf{y}\| = \sum_{k=1}^{n} \frac{1}{k}|x(k) + y(k)| \leqslant \sum_{k=1}^{n} \frac{1}{k}|x(k)| + \sum_{k=1}^{n} \frac{1}{k}|y(k)| = \|\mathbf{x}\| + \|\mathbf{y}\|$.

Hence the given set is the closed unit ball in this normed vector space and so is certainly a closed set. Since all norms on $\mathbb{C}^n$ are equivalent, the set is also closed for the original norm.

[Note: this last exercise is likely to be done in two or three different ways, each of similar length.]