PMA345 Functional Analysis June 2002 — Solutions

1. (i) [4 marks: bookwork definitions] *Define carefully what is meant by a normed space.*

Let E be a vector space over $\mathbb{C}$. By a norm on E is meant a function $x \mapsto \|x\|$ mapping E into $\mathbb{R}^+$ for which, for $x, y \in E$ and $\lambda \in \mathbb{C}$:

(a) $\|x+y\| \leqslant \|x\| + \|y\|$;

(b) $\|\lambda x\| = |\lambda|\,\|x\|$;

(c) $\|x\| = 0 \iff x = 0$.

We then say that $(E, \|.\|)$ is a normed space.

1. (ii)(a) [3 marks: bookwork] *Let E be a normed space. Prove that if the sequences (x_n) and (y_n) in E converge to x, y, respectively, then $(x_n + y_n)$ converges to $x + y$.*

If $x_n \to x$ and $y_n \to y$, then $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$, so

$$\begin{aligned} \|(x_n + y_n) - (x + y)\| &= \|(x_n - x) + (y_n - y)\| \\ &\leqslant \|x_n - x\| + \|y_n - y\| \\ \to 0 + 0 = 0, & \end{aligned}$$

as $n \to \infty$. Therefore $x_n + y_n \to x + y$.

1. (ii)(b) [1 mark: bookwork] *State, without proof, a similar theorem concerning scalar multiplication.*

If $x_n \to x$ in E and $\lambda_n \to \lambda$ in $\mathbb{C}$, then $\lambda_n x_n \to \lambda x$.

1. (ii)(c) [5 marks: unseen problem] *A set $C \subseteq E$ is said to be* convex *if $\lambda x + \mu y \in C$ whenever $x, y \in C$ and $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$. Show that if a, b are any two points in E, and $R > 0$ then the set*

$$\{x \in E : \|x - a\| + \|x - b\| < R\}$$

is convex.

Let

$$C = \{x \in E : \|x - a\| + \|x - b\| < R\}.$$

Suppose $x, y \in C$ and $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$; we must show that $\lambda x + \mu y \in C$:

$$\begin{aligned} \|(\lambda x + \mu y) - a\| + \|(\lambda x + \mu y) - b\| &= \|\lambda(x - a) + \mu(y - a)\| + \|\lambda(x - b) + \mu(y - b)\| \\ &\leqslant \lambda\|x - a\| + \mu\|y - a\| + \lambda\|x - a\| + \mu\|y - a\| \\ &< (\lambda + \mu)R \\ &= R. \end{aligned}$$

1. (iii)(a) [5 marks: bookwork] *Let E be a normed space. Prove that if $x, y \in E$ then*

$$|\,\|x\| - \|y\|\,| \leqslant \|x - y\|,$$

and deduce that the map $x \to \|x\|$ of E to $\mathbb{R}$ is continuous.

We observe that

$$\|x\| = \|(x - y) + y\| \leqslant \|x - y\| + \|y\|$$

so

$$\|x\| - \|y\| \leqslant \|x - y\|.$$

Therefore

$$\|y\| - \|x\| \geqslant -\|x - y\| = -\|y - x\|. \tag{1}$$

Interchanging the rôles of x and y in (1):

$$\|x\| - \|y\| \geqslant -\|x - y\| \qquad . \tag{2}$$

Combining (1) and (2):

$$-\|x-y\| \leqslant \|x\| - \|y\| \leqslant \|x-y\|,$$

that is,

$$|\,\|x\| - \|y\|\,| \leqslant \|x-y\|.$$

Therefore $\|.\| : E \to \mathbb{C}$ is continuous: just take $\delta = \varepsilon$ in the characterization of continuity.

1. (iii)(b) [2 marks: unseen problem] *Prove that the set*

$$S = \{x \in E : \|x\| = 1\}$$

is closed in E.

The set S is the inverse image of the closed set $\{1\} \subseteq \mathbb{R}$ under the continuous map $x \mapsto \|x\|$, and is therefore closed.

1. (iii)(c) [5 marks: unseen problem] *Without assuming the normed space E to be complete, show that if (x_n) is a Cauchy sequence in E, then then $\|x_n\| \to R$ for some $R \in \mathbb{R}$.*

Since (x_n) is Cauchy, we know that for all $\varepsilon > 0$ there exists N such that for all $p, q \geqslant N$,

$$\|x_p - x_q\| < \varepsilon.$$

By (iii)(a), it follows that

$$|\,\|x_p\| - \|x_q\|\,| < \varepsilon.$$

Thus the sequence $(\|x_n\|)$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, this sequence is convergent in $\mathbb{R}$.

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2. (i) [11 marks: bookwork] *Let E, F, G be normed spaces. Prove that a linear map T of E into F is continuous if and only if there is a number $K \geqslant 0$ such that $\|Tx\| \leqslant K\|x\|$ $(x \in E)$.*

Suppose $\|Tx\| \leqslant K\|x\|$ $(x \in E)$; then if $x_n \to x_0$ in E, we have

$$\begin{aligned} \|Tx_n - Tx_0\| &= \|T(x_n - x_0)\| \\ &\leqslant K\|x_n - x_0\| \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

so T is continuous.

For the converse, suppose T is not bounded; then, for all n there exists $x_n \in E$ with

$$\|Tx_n\| > n\|x_n\|.$$

We must have $x_n \neq 0$ (otherwise $\|Tx_n\| = 0 = n\|x_n\|$) so, writing

$$y_n = \frac{1}{\sqrt{n}\,\|x_n\|} x_n,$$

we have

$$\|y_n\| = \left| \frac{1}{\sqrt{n}\,\|x_n\|} \right| \|x_n\| = \frac{1}{\sqrt{n}}$$

so $y_n \to 0$, and

$$\begin{aligned} \|Ty_n\| &= \left\| \frac{1}{\sqrt{n}\,\|x_n\|} T(x_n) \right\| \\ &= \left| \frac{1}{\sqrt{n}\,\|x_n\|} \right| \|T(x_n)\| \\ &> \left| \frac{1}{\sqrt{n}\,\|x_n\|} \right| n\,\|x_n\| \\ &= \sqrt{n} \end{aligned}$$

so $Ty_n \not\to 0$. Thus T is not continuous.

2. (ii) [2 marks: bookwork definition] *Define the norm, $\|T\|$, of a bounded linear map of E into F.*
We define $\|T\|$ to be the least value of K such that $\|Tx\| \leqslant K\|x\|$ $(x \in E)$.

2. (iii) [6 marks: modified bookwork] *Let $\mathcal{B}(E,F)$ be the vector space of all bounded linear mappings of E into F. Show that $\|.\|$ as defined in (ii) is a norm on $\mathcal{B}(E,F)$. (You may assume that $\mathcal{B}(E,F)$ is a vector space.)*

1. If $S, T \in \mathcal{B}(E,F)$, then for all $x \in E$,

$$\begin{aligned} \|(S+T)(x)\| &= \|Sx + Tx\| \\ &\leqslant \|Sx\| + \|Tx\| \\ &\leqslant \|S\|\,\|x\| + \|T\|\,\|x\| \\ &\leqslant (\|S\| + \|T\|)\|x\|. \end{aligned}$$

   Therefore $\|S+T\| \leqslant \|S\| + \|T\|$.

2. If $T \in \mathcal{B}(E,F)$ and $\lambda \in \mathbb{C}$, then for all $x \in E$,

$$\|(\lambda T)(x)\| = \|\lambda(Tx)\| = |\lambda|\,\|Tx\| \leqslant |\lambda|\,\|T\|\,\|x\|.$$

   Therefore

$$\|\lambda T\| \leqslant |\lambda|\,\|T\|. \tag{3}$$

   If $\lambda = 0$, we have $\|\lambda T\| = \|0\| = 0 = |\lambda|\,\|T\|$. If $\lambda \neq 0$, we use (3) with $1/\lambda$ in place of λ to obtain

$$|\lambda|\,\|T\| = |\lambda| \left\| \frac{1}{\lambda}(\lambda T) \right\| = |\lambda| \left| \frac{1}{\lambda} \right| \|\lambda T\| = \|\lambda T\|.$$

   Combining this with (3) shows $\|\lambda T\| = |\lambda|\,\|T\|$.

3. Finally,

$$\begin{aligned} \|T\| = 0 &\iff \|Tx\| \leqslant 0\|x\| \qquad (x \in E) \\ &\iff \|Tx\| = 0 \qquad (x \in E) \\ &\iff Tx = 0 \qquad (x \in E) \\ &\iff T = 0. \end{aligned}$$

2. (iv) [3 marks: homework problem] *Let $S \in \mathcal{B}(E,F)$ and $T \in \mathcal{B}(F,G)$. Let $T \circ S$ be the linear mapping defined by $T \circ Sx = T(Sx)$ $(x \in E)$. Show that $T \circ S$ is bounded with*

$$\|T \circ S\| \leqslant \|T\|\,\|S\|.$$

By the definition of the operator norm, for all $x \in E$,

$$\begin{aligned} \|T \circ Sx\| &= \|T(Sx)\| \\ &\leqslant \|T\|\,\|Sx\| \\ &\leqslant \|T\|(\|S\|\,\|x\|) \\ &\leqslant (\|T\|\,\|S\|)\|x\|, \end{aligned}$$

so $T \circ S$ is bounded with $\|T \circ S\| \leqslant \|T\|\,\|S\|$.

2. (v) [3 marks: unseen problem] *If $T \in \mathcal{B}(E,F)$ and $S \in \mathcal{B}(F,E)$ are such that $T \circ S$ is the identity operator on F, what relation can you deduce between $\|T\|$ and $\|S\|$?*

The norm of the identity operator I_F must be 1, since $\|I_F x\| = \|x\|$ $(x \in F)$. Therefore

$$\|T\|\,\|S\| \geqslant \|T \circ S\| = 1,$$

so

$$\|T\| \geqslant \|S\|^{-1}.$$

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3. (i) [3 marks: bookwork] *State the Hahn-Banach Theorem for (real and complex) normed spaces.*

Let $(E, \|.\|)$ be either a (complex) normed space or a real normed space, and let M be a vector subspace of E. Let f be a continuous linear functional on M. Then there is a continuous linear functional $\overline{f}$ on E extending f with $\|\overline{f}\| = \|f\|$.

3. (ii) [14 marks: bookwork] *Prove the Hahn-Banach Theorem for complex normed spaces, assuming that it holds for real normed spaces.*

Given a continuous (complex) linear functional f on M, we write

$$f(x) = f_1(x) + i f_2(x) \qquad (x \in M),$$

where $f_1, f_2 : M \to \mathbb{R}$ are continuous real linear functionals on M, considered as a real normed space. Since $|f_1(x)| \leqslant |f(x)|$, we have $\|f_1\| \leqslant \|f\|$. Now f was linear over the complex field so

$$f(ix) = i f(x) \qquad (x \in M).$$

That is,

$$f_1(ix) + i f_2(ix) = i f_1(x) - f_2(x) \qquad (x \in M).$$

Equating real parts:

$$f_2(x) = -f_1(ix) \qquad (x \in M). \tag{4}$$

Now $(E, \|.\|)$ may be viewed as a real normed space, so we may apply the real case (which is given) to deduce that there is a continuous real linear functional $\overline{f_1}$ on E extending f_1, with $\|\overline{f_1}\| = \|f_1\| \leqslant \|f\|$. We define $\overline{f_2}$ on E by

$$\overline{f_2}(x) = -\overline{f_1}(ix) \qquad (x \in E).$$

Then, by (4), $\overline{f_2}$ is an extension of f_2 to E. Let

$$\overline{f}(x) = \overline{f_1}(x) + i\overline{f_2}(x) \qquad (x \in E).$$

Then not only is $\overline{f}$ a continuous real linear functional on E, but

$$\begin{aligned} \overline{f}(ix) &= \overline{f_1}(ix) + i\overline{f_2}(ix) \\ &= -\overline{f_2}(x) - i\overline{f_1}(-x) \\ &= i\overline{f_1}(x) - \overline{f_2}(x) \\ &= i\overline{f}(x), \end{aligned}$$

so $\overline{f}$ a complex linear functional on E. To find $\|\overline{f}\|$, suppose $x \in E$ and $\overline{f}(x) = re^{i\theta}$ in modulus–argument form: then $\overline{f}(e^{-i\theta}x) = r$ which is real, so $\overline{f_1}(e^{-i\theta}x) = r$. Therefore

$$|\overline{f}(x)| = r = \left|\overline{f_1}\left(e^{-i\theta}x\right)\right| \leqslant \|f\|\,\left\|e^{-i\theta}x\right\| = \|f\|\,\|x\|,$$

so $\|\overline{f}\| \leqslant \|f\|$.

For the reverse inequality we have

$$\begin{aligned} \|\overline{f}\| &= \sup\{|\overline{f}(x)| : x \in E\} \\ &\geqslant \sup\{|\overline{f}(x)| : x \in M\} \\ &= \sup\{|f(x)| : x \in M\} \\ &= \|f\|, \end{aligned}$$

so the desired result follows.

3. (iii) [8 marks: bookwork] *Use the Hahn-Banach theorem to show that for every $x \in E$,*

$$\|x\| = \sup\{|f(x)| : f \in E',\ \|f\| \leqslant 1\}.$$

By definition of $\|f\|$, we have $|f(x)| \leqslant \|x\|$ whenever $f \in E'$ with $\|f\| \leqslant 1$, so

$$\|x\| \geqslant \sup\{|f(x)| : f \in E',\ \|f\| \leqslant 1\}.$$

To obtain the reverse inequality, we use the Hahn–Banach Theorem. If $x \in E$ and $x \neq 0$ (note that the inequality is trivial if $x = 0$), then we apply the Hahn–Banach Theorem with $M = \operatorname{span}(x)$. We define f on M by

$$f(\lambda x) = \lambda \|x\|,$$

so that $f(x) = \|x\|$ and

$$\|f\| = \sup_{\lambda \neq 0} \frac{|\lambda|\, \|x\|}{\|\lambda x\|} = 1.$$

The Hahn–Banach Theorem then allows us to extend f to a continuous linear functional, which we shall also denote by f on the whole of E with these same properties. Then

$$\|f\| = 1 \quad \text{and} \quad |f(x)| = \|x\|,$$

so

$$\|x\| \leqslant \sup\{|f(x)| : f \in E',\ \|f\| \leqslant 1\}.$$

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4. (i) [17 marks: bookwork] *Let E, F be normed spaces, and $L \subseteq \mathcal{B}(E, F)$. What does it mean to say that L is (a)* uniformly bounded*; (b)* pointwise bounded*? State and prove the Uniform Boundedness Theorem on the relation between these two concepts under suitable conditions.*

The set L is *uniformly bounded* if there exists $M > 0$ such that $\|T\| \leqslant M$ for all $T \in L$. The set L is *pointwise bounded* if, for all $x \in E$ there exists $M_x > 0$ such that $\|Tx\| \leqslant M_x$ for all $T \in L$.

Theorem. *[Uniform Boundedness Theorem] If E, F be normed spaces, and $L \subseteq \mathcal{B}(E, F)$, and E is complete, then L is uniformly bounded if and only if it is pointwise bounded.*

Proof. The direction 'uniformly bounded implies pointwise bounded' is trivial: if $\|T\| \leqslant M$ for all $T \in L$, then $\|Tx\| \leqslant M\|x\|$ and $M_x = M\|x\|$ is the desired pointwise bound.

For the converse, let

$$B = \{x \in E : \|Tx\| \leqslant 1 \quad (T \in L)\}.$$

The fact that L is pointwise bounded implies that for all $x \in E$ there is a positive integer n such that

$$\|Tx\| \leqslant n \qquad (T \in L).$$

Therefore, for all $x \in E$, there exists n such that $n^{-1}x \in B$. That is,

$$E = \bigcup_{n=1}^{\infty} nB.$$

Now

$$B = \bigcap_{T \in L} T^{-1}(F_1).$$

Since the maps T are continuous, the sets $T^{-1}(F_1)$ are closed, so the set B is closed, so (since the maps $x \mapsto nx$ are homeomorphisms), the sets nB are all closed.

Thus we have the complete metric space E expressed as a countable union of closed sets. Baire's Category Theorem implies that at least one of these sets must have non-empty interior: say $\operatorname{int}(n_0 B) \neq \emptyset$. Since $x \mapsto n_0 x$ is a homeomorphism, $\operatorname{int}(B) \neq \emptyset$.

Therefore there exists $x_0 \in B$ and $\delta > 0$ such that

$$B(x_0; \delta) \subseteq B;$$

i.e., for all $x \in E_\delta^\circ$,

$$x_0 + x \in B \qquad \text{and} \qquad x_0 \in B;$$

i.e., for all $x \in E_\delta^\circ$ and all $T \in L$,

$$\|T(x_0 + x)\| \leqslant 1 \qquad \text{and} \qquad \|T(x_0)\| \leqslant 1;$$

so

$$\|T(x)\| \leqslant \|T(x_0 + x)\| + \|-T(x_0)\| \leqslant 1 + 1 = 2.$$

Therefore

$$\|Tx\| \leqslant \frac{2}{\delta} \qquad (\|x\| < 1,\ T \in L).$$

So $\|T\| \leqslant 2/\delta$ $(T \in L)$. So L is uniformly bounded. $\Diamond$

4. (ii) [8 marks: unseen problem] *Let $E = F = c_{00}$, the space of all eventually zero sequences*

$$\mathbf{x} = (x_1, x_2, x_3, \ldots)$$

with the supremum norm

$$\|\mathbf{x}\| = \sup\{|x_i| : i = 1, 2, 3, \ldots\}.$$

Show that the linear mappings T_n $(n = 1, 2, 3, \ldots)$ defined by

$$T_n(\mathbf{x}) = (x_1, 2x_2, 3x_3, \ldots, nx_n, 0, 0, 0, \ldots) \qquad (\mathbf{x} \in c_{00})$$

are bounded on c_{00}, i.e. are elements of $\mathcal{B}(E, F)$, and that the set $L = \{T_n : n = 1, 2, 3, \ldots\}$ is pointwise bounded but not uniformly bounded. Why does this not contradict the Uniform Boundedness Theorem?

If

$$\mathbf{x} = (x_1, x_2, x_3, \ldots),$$

then

$$\begin{aligned} \|T_n \boldsymbol{x}\| &= \|(x_1, 2x_2, 3x_3, \ldots, nx_n, 0, 0, 0, \ldots)\| \\ &= \sup\{i|x_i| : 1 \leqslant i \leqslant n\} \\ &\leqslant n \sup\{|x_i| : 1 \leqslant i \leqslant n\} \\ &\leqslant n\|\boldsymbol{x}\|. \end{aligned}$$

Thus T_n is bounded, (with $\|T_n\| \leqslant n$).

To show that L is pointwise bounded, we consider the action of the T_n on a fixed element

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_k, 0, 0, 0, \ldots) \in c_{00}.$$

Then if $n \geqslant k$,

$$\begin{aligned} \|T_n \boldsymbol{x}\| &= \|(x_1, 2x_2, 3x_3, \ldots, kx_k, 0, 0, 0, \ldots)\| \\ &= \sup\{i|x_i| : 1 \leqslant i \leqslant k\} \\ &\leqslant k \sup\{|x_i| : 1 \leqslant i \leqslant k\} \\ &\leqslant k\|\boldsymbol{x}\|. \end{aligned}$$

Thus L is pointwise bounded, (with $M_x = k$ in the notation above).

The set L is not uniformly bounded since $\|T_n\| \geqslant n$. To show this, let $\boldsymbol{x}$ be defined by

$$x_i = \begin{cases} 1 & (i = n) \\ 0 & (i \neq n). \end{cases}$$

Then $\|\boldsymbol{x}\| = 1$, but $\|T_n\boldsymbol{x}\| \geqslant |(T_n\boldsymbol{x})_n| = n$.

This does not contradict the Uniform Boundedness Principle because the space $E = c_{00}$ is not complete.

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5. (i) [13 marks: bookwork] *Let E be a normed space of finite dimension n. Prove that*

(a) all linear functionals on E are continuous;

(b) E is isomorphic to the space $(\mathbb{C}^n, \|.\|_\infty)$;

(c) E is complete.

We prove this by induction on n.

For $n = 0$ we have $E = \{0\}$ and the results are trivial.

Suppose the theorem holds for dimension $n-1$ and $\dim E = n$. If f is a non-zero linear functional on E, then $\ker f$ is a linear subspace of E of dimension $n-1$. Applying the induction hypothesis to $\ker f$, we obtain that $\ker f$ is complete. Therefore $\ker f$ is closed in E. Therefore f is continuous, which proves (a) for dimension n.

Now (a) implies (b), for if we choose a basis $\{x_1, \ldots, x_n\}$ of E, then for all $x \in E$ we can write

$$x = f_1(x)x_1 + \ldots + f_n(x)x_n,$$

where the f_i are uniquely defined, linear functionals. By (a), the f_i are continuous. Define

$$T : x \mapsto (f_1(x), \ldots, f_n(x)) : (E, \|.\|) \to (\mathbb{C}^n, \|.\|_\infty).$$

Then T is continuous, since all the f_i are continuous. But

$$T^{-1} : (\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 x_1 + \ldots + \lambda_n x_n$$

is continuous, by the continuity of algebraic operations in normed spaces. Therefore T is an isomorphism.

Further, (b) implies (c), since $(\mathbb{C}^n, \|.\|_\infty)$ is complete and isomorphisms preserve completeness.

Thus we have deduced from the induction hypothesis that (a), (b), and (c) hold for dimension n. The desired result follows by induction.

5. (ii) [6 marks: unseen problem] *Let P_k be the set of all polynomials $p(t)$ of degree at most k with complex coefficients. By considering linear functionals on $(P_k, \|.\|)$ for a suitable norm $\|.\|$, or otherwise, show that, for every $t \in [0,1]$ there is a constant K, depending on t and k but not on p, such that*

$$|p(t)| \leqslant K \int_0^1 |p(s)|\, ds \qquad (p \in P_k).$$

The vector space P_k is finite-dimensional (dimension k). For each $t \in [0,1]$, let f_t be the linear functional on P_k defined by

$$f_t(p) = p(t) \qquad (p \in P_k).$$

Let P_k be given the norm

$$\|p\| = \int_0^1 |p(s)|\, ds \qquad (p \in P_k).$$

The theorem proved above tells us that the linear functionals f_t are bounded on $(P_k, \|.\|)$; i.e.

$$|p(t)| \leqslant K \int_0^1 |p(s)|\, ds \qquad (p \in P_k),$$

where $K = \|f_t\|$.

5. (iii) [6 marks: unseen problem] *By considering suitable polynomials p, or otherwise, show that it is not possible to find K independent of k such that the above inequality holds for all p.*

Let $p_k(t) = t^k \in P_k$. Then

$$\int_0^1 |p_k(s)|\, ds = \int_0^1 s^k\, ds = \left[\frac{s^{k+1}}{k+1}\right]_0^1 = \frac{1}{k+1}.$$

Now take $t = 1$, and we have

$$|p_k(t)| = (k+1)\int_0^1 |p_k(s)|\, ds,$$

so no K can exist independent of k.