PMA445 Functional Analysis 2003–04

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1 Introduction

1.1 Administration

The course is two lectures per week, Tuesdays at 9.00 and Thursdays at 3.10, both in Lecture Room 5. Set work will be given approximately once a week and solutions distributed one week later; set work will be collected and marked approximately one week in three.

1.2 Revision Chapters

Certain chapters — those on Countability, Metric Spaces and Compactness — merely record results which you will have met, in more or less this form, in PMA215 and PMA344. These are included so that the results can easily be referred to later, but they are not directly examinable in the examination on this course. That is, you will not be asked to prove the results contained in these chapters as bookwork. You will, however, need to be able to use the results in bookwork and problem questions based on the remaining chapters.

1.3 Books

1. C. Goffman and G Pedrick "First Course in Functional Analysis", Chelsea (1983) (Available in Blackwells £21).
2. G.F. Simmons, "Introduction to Topology and Modern Analysis", Kreiger (1999), £40.50 (paperback version temporarily(?) out of print).
3. W. Rudin, "Real and Complex Analysis", McGraw-Hill (1966).
4. N.J. Young "An Introduction to Hilbert Space", Cambridge (1988).

1.4 Course Description

Many topics in real and complex analysis can be usefully approached by thinking of functions as points in an infinite-dimensional space: a function $f : \mathbb{R} \to \mathbb{R}$ having one coordinate $f(x)$ for each $x \in \mathbb{R}$. For example, with this view, expressing a function as a Fourier series corresponds to writing its position vector as a linear combination of basis vectors, where the (orthogonal) basis consists of the functions $\sin nx$ $(n = 1, 2, 3, \ldots)$ and $\cos nx$ $(n = 0, 1, 2, 3, \ldots)$; (see the Linear Mathematics II course).

You have seen other examples in the Metric Spaces and Real Analysis courses: examples with a more analytic flavour. The chief characteristic of Functional Analysis is that it merges the analytic and the algebraic: so the examples we shall be considering will, like the Fourier Series example above, rely on considering functions as points in a vector space rather than just a metric space. However, the vector space will always have a metric structure, so we shall have (linear) algebra and analysis working together.

One can study just vector spaces with a metric, together with axioms connecting the two — for example, saying that the algebraic operations are all continuous. However, this level of generality is unnecessary in a first course. A simpler and very useful theory is obtained by requiring a rather special metric: by studying **normed spaces**.

The course will begin by introducing normed vector spaces and their dual spaces, look at some examples, and then establish, amongst other things, two of basic results of the theory: the Hahn-Banach Theorem and the Uniform Boundedness Theorem. The latter is a consequence of Baire's Category Theorem — one of the key theorems of the Real Analysis course.

2 Metric spaces (revision) and products of metric spaces (new)

Basic idea: much of real analysis can be done in terms of the distance function

$$\text{distance}(x, y) = |x - y| \quad (x, y \in \mathbb{R}).$$

We abstract the properties of this function needed for analysis.

Definition 2.1 A *metric* on a set X is a function $d : X \times X \to \mathbb{R}^+$ such that

1. $d(x, z) \leqslant d(x, y) + d(y, z)$ $(x, y, z \in X)$,
2. $d(x, y) = d(y, x)$ $(x, y \in X)$,
3. $d(x, y) = 0$ iff $x = y$ $(x, y \in X)$.

A *metric space* is a pair (X, d) consisting of a set X and a metric d on X.

Example 2.2 1. The real line $\mathbb{R}$, with metric $d(x, y) = |x - y|$.

2. The real plane $\mathbb{R}^2$, with metric

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}.$$

Many of the notions associated with real analysis have generalizations to metric spaces.

Definition 2.3 A sequence (x_n) in a metric space X (strictly, (X, d)) *converges* to a point $x \in X$ iff for all $\varepsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that for all $n \geqslant N$, $d(x_n, x) < \varepsilon$.

Definition 2.4 A subset A of a metric space X is *open* if, for all $a \in A$, there exists $\varepsilon > 0$ such that

$$B(a; \varepsilon) := \{x \in X : d(x, a) < \varepsilon\} \subseteq A.$$

Example 2.5 Every open ball $B(x; \varepsilon)$ in a metric space is an open set.

Definition 2.6 A subset A of a metric space X is *closed* if whenever a sequence (a_n) in A converges to a point $x \in X$, we have $x \in A$.

Example 2.7 In $\mathbb{R}$ the set $[0, 1]$ is closed, but the set $(0, 1]$ is neither open nor closed. Typically, most subsets of a metric space are neither open nor closed.

A set is closed if and only if its complement is open; consequently, the whole theory of closed sets is a reflection of that of open sets, with unions becoming intersections and vice versa, subsets becoming supersets, *et cetera* .

Definition 2.8 The *closure* of a set A in a metric space X is the set

$$\bar{A} := \{x \in X : \forall \varepsilon > 0 \quad B(x; \varepsilon) \cap A \neq \emptyset\}.$$

Equivalently, $\bar{A}$ is the smallest closed set containing A.

Definition 2.9 A set D is said to be *dense* in a metric space X if $\bar{D} = X$.

Definition 2.10 [new] A metric space is *separable* if it has a countable dense subset.

Definition 2.11 If $(X, d_1), (Y, d_2)$, are two metric spaces, then a function $f : X \to Y$ is said to be *continuous* at a point $x_0 \in X$ iff $f(x_n) \to f(x_0)$ in Y whenever $x_n \to x_0$ in X.

If f is continuous at every point, we say that f*is continuous.*

Definition 2.12 [new] A *homeomorphism* is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous. We say that X and Y are *homeomorphic* if there is a homeomorphism $f : X \to Y$.

Homeomorphisms preserve most of the important properties of metric spaces except completeness and total boundedness (see later for definitions). They preserve all properties definable in terms of convergent sequences (but not Cauchy sequences).

We can prove many real analysis theorems just as easily in the context of metric spaces. For example:

Proposition 2.13 *In a metric space X:*

1. *the sets $\emptyset$ and X are open and closed;*
2. *the intersection of two open sets is open: the union of two closed sets is closed;*
3. *arbitrary unions of open sets are open: arbitrary intersections of closed sets are closed.*

Proposition 2.14 *For a function $f : X \to Y$ between metric spaces, the following are equivalent:*

1. *f is continuous;*
2. *for all $x_0 \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d_2(f(x), f(x_0)) < \varepsilon$ whenever $x \in X$ with $d_1(x_0, x) < \delta$;*
3. *$f^{-1}(G)$ is open in X, for every open set $G \subseteq Y$;*
4. *$f^{-1}(F)$ is closed in X, for every closed set $F \subseteq Y$.*

Definition 2.15 [new] The *product* of two metric spaces (X_1, d_1) and (X_2, d_2) is the space

$$X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$$

with the metric

$$d((a_1, a_2), (b_1, b_2)) = \max\{d_1(a_1, b_1), d_2(a_2, b_2)\}.$$

Exercise 2.16 Two alternative metrics on $X_1 \times X_2$ are the "taxi-cab metric"

$$d'((a_1, a_2), (b_1, b_2)) = d_1(a_1, b_1) + d_2(a_2, b_2)$$

and

$$d''((a_1, a_2), (b_1, b_2)) = \sqrt{d_1(a_1, b_1)^2 + d_2(a_2, b_2)^2}.$$

Show that, for $a, b \in X_1 \times X_2$,

$$d(a, b) \leqslant d''(a, b) \leqslant d'(a, b) \leqslant 2d(a, b).$$

Definition 2.17 A sequence (x_n) in a metric space (X, d) is said to be *Cauchy* if, for all $\varepsilon > 0$ there exists a positive integer N such that $d(x_p, x_q) < \varepsilon$ for all $p, q \geqslant N$. Informally, one may express this as '$d(x_p, x_q) \to 0$ as $p, q \to \infty$'.

It is easy to show that every convergent sequence is Cauchy; we are particularly fond of metric spaces in which the converse holds.

Definition 2.18 A metric space (X, d) is *complete* if every Cauchy sequence in X converges (in X).

Examples 2.19

1. The set $\mathbb{R}$ in the usual metric is a complete metric space.
2. The set $\mathbb{Q}$ (the rationals) in the usual metric is not complete, because a sequence $x_n \in \mathbb{Q}$ with $x_n \to \sqrt{2}$ in $\mathbb{R}$ is convergent in $\mathbb{R}$, so Cauchy in $\mathbb{R}$, so Cauchy in $\mathbb{Q}$, but is not convergent in $\mathbb{Q}$.

3 Compactness, Sups and Infs (mainly revision)

The main topic of this chapter is the related notion of 'compactness'.

Definition 3.1 A subset K of a metric space (X, d) is *compact* if every sequence in K has a subsequence convergent in (X, d) to a point of K.

Examples 3.2

1. The real line $\mathbb{R}$ with the usual metric is not compact (as a subset of itself), because the sequence $x_n = n$ $(n = 1, 2, 3, \ldots)$ does not have a convergent subsequence.

2. The set $(0, 1) \subseteq \mathbb{R}$ is not compact, because the sequence $x_n = 1 - \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$ does not have a subsequence convergent *in the set* $(0, 1)$; every subsequence converges to 1, which is not in $(0, 1)$, and therefore cannot converge to anything in $(0, 1)$.

3. The set $[0, 1] \subseteq \mathbb{R}$ is compact.

   Sketch proof

   Let (x_n) be any sequence in $[0, 1]$; we shall construct a convergent subsequence (x_{n_i}). Let $n_1 = 1$. Now, either there are infinitely many n for which x_n is in $[0, 1/2]$ or there are infinitely many in $[1/2, 1]$, (or both). Suppose the former. Let n_2 be any number greater than n_1 such that $x_{n_2} \in [0, 1/2]$. Now repeat the process on the interval $[0, 1/2]$: either there are infinitely many n with $x_n \in [0, 1/4]$ or there are infinitely many with $x_n \in [1/4, 1/2]$; if, say, the latter, then choose $n_3 > n_2$ such that $x_{n_3} \in [1/4, 1/2]$ and then proceed to consider the interval $[1/4, 1/2]$. In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ such that

   $$|x_{n_p} - x_{n_q}| \leqslant 2^{-N} \qquad (p, q, > N).$$

   It follows that this subsequence is Cauchy, and so convergent.

4. The set $[0, 1]^n \subseteq \mathbb{R}^n$ is compact. The proof is the same as for the $n = 1$ case above, except that instead of bisecting the interval at each stage, we split it into 2^n cubes by bisecting in each coordinate direction.

Theorem 3.3 ((1) and (4) new?)

1. *The intersection of a closed set and a compact set is always compact.*

2. *Every compact subset of a metric space is closed.*

3. *Continuous images of compact sets are compact; i.e. if $f : X \to Y$ is a continuous mapping between metric spaces and $K \subseteq X$ is compact, then $f(K)$ is a compact subset of Y.*

4. *Finite unions of compact sets are compact*

Proof.

1. Let $K \subseteq X$ be a compact set and $F \subseteq X$ a closed subset. To show that $K \cap F$ is compact, we suppose (x_n) is a sequence in $K \cap F$ and we show that (x_n) has a subsequence convergent *in the set* $K \cap F$. Since K is compact, there is a subsequence (x_{n_i}) convergent to some $x \in K$. Now F is closed and $x_{n_i} \in F$, so $x \in F$. Therefore (x_{n_i}) is convergent to a point in $K \cap F$, as desired.

2. Let K be a compact subset of a metric space X. To show that K is closed, we suppose (x_n) is a sequence in K with $x_n \to x \in X$. Since K is compact, (x_n) has a subsequence (x_{n_i}) convergent to some $y \in K$. However, subsequences of convergent sequences are convergent to the same limits, so $x_{n_i} \to x$. Therefore $x = y \in K$. This proves that K is closed.

3. Let $f : X \to Y$ be a continuous mapping between metric spaces and let $K \subseteq X$ be compact. To show that $f(K)$ is compact, suppose (y_n) is a sequence in $f(K)$. Then $y_n = f(x_n)$ for some $x_n \in K$, $(n = 1, 2, 3, \ldots)$. Since K is compact, the sequence (x_n) has a convergent subsequence $x_{n_i} \to x \in K$. Since f is continuous, it follows that

   $$y_{n_i} = f(x_{n_i}) \to f(x) \in f(K).$$

   Thus $f(K)$ is compact.

4. We show that the union of two compact sets is always compact, and then the result for finite unions follows by an easy induction. Let $K, L \subseteq X$ be compact, and let (x_n) be a sequence in $K \cup L$. Then either $x_n \in K$ for infinitely many n or $x_n \in L$ for infinitely many n (or both). If the former, then, by the compactness of K, there is a subsequence (x_{n_i}) which is convergent to a point which is in K and so in $K \cup L$; if the latter, the compactness of L yields the desired subsequence in the same way.

$\Diamond$

Exercise 3.4 Show that every decreasing sequence $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ of compact non-empty sets in a metric space has a non-empty intersection.

Exercise 3.5 Show that if $K_1 \subseteq X_1$ and $K_2 \subseteq X_2$ are compact subsets of metric spaces X_1, X_2, then $K_1 \times K_2$ is a compact subset of the metric space $X_1 \times X_2$.

In $\mathbb{R}^n$, the compact sets are easily characterized.

Theorem 3.6 (Heine–Borel in $\mathbb{R}^n$; PMA344 had the $n = 1$ case) *A set $K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Suppose $K \subseteq \mathbb{R}^n$ is compact. It follows from Theorem 3.3(2) that it is closed. If it were unbounded, we could find a sequence (x_n) in K such that $|x_n| \geqslant n$ $(n = 1, 2, 3, \ldots)$. If this sequence had a convergent subsequence $x_{n_i} \to x$, say, then we should have $|x_{n_i}| \to |x| < \infty$, which is impossible.

Conversely, suppose $K \subseteq \mathbb{R}^n$ is closed and bounded. Then $K \subseteq [-R, R]^n$ for some sufficiently large number R. Now there is a continuous surjective map

$$f : (x_1, \ldots, x_n) \mapsto (2Rx_1 - R, \ldots, 2Rx_n - R) : [0, 1]^n \to [-R, R]^n,$$

so, by Theorem 3.3(3), $[-R, R]^n$ is compact. By Theorem 3.3(1), $K = K \cap [-R, R]^n$ is compact. $\Diamond$

Let us consider the extension of Theorem 3.6 to general metric spaces. We used the completeness of $\mathbb{R}^n$, (in the proof that $[0, 1]^n$ is compact), so we expect to work in complete metric spaces. The notion of "boundedness" can be extended straightforwardly to general metric spaces, but it turns out that the correct strategy is to replace it by the notion of "total boundedness" (sometimes called "precompactness"), which is a property of bounded sets akin to that used in the repeated bisection argument.

Definition 3.7 A set K in a metric space (X, d) is said to be *totally bounded* if, for every $\varepsilon > 0$, there is a finite set $\{x_1, x_2, \ldots, x_n\} \subseteq K$ such that

$$K \subseteq B(x_1; \varepsilon) \cup B(x_2; \varepsilon) \cup \ldots \cup B(x_n; \varepsilon).$$

Exercise 3.8 Show that if K is totally bounded, the x_i in the definition may be taken to lie anywhere in X. (Hint: use the x_i in X for $\varepsilon/2$ to get the desired x_i in K for ε.)

Theorem 3.9 (essentially Theorem 5.9 of PMA344) *A subset K of a complete metric space X is compact if and only if it is closed and totally bounded.*

Proof. We shall not need the 'if' part of this theorem, so we omit the proof. For the 'only if' part, we need to show that if K is compact, then it is totally bounded.

Suppose that K is a compact subset of a complete metric space and that K is not totally bounded. Thus there exists $\varepsilon > 0$ such that for no finite set $\{x_1, \ldots, x_n\} \subseteq K$ do we have

$$K \subseteq B(x_1; \varepsilon) \cup B(x_2; \varepsilon) \cup \ldots \cup B(x_n; \varepsilon).$$

Thus, for every finite set $\{x_1, \ldots, x_n\} \subseteq K$ there is an element $x_{n+1} \in K$ with $d(x_{n+1}, x_i) \geqslant \varepsilon$ $(1 \leqslant i \leqslant n)$. Using this inductively, we can find a sequence $(x_n) \subseteq K$ such that $d(x_i, x_j) \geqslant \varepsilon$ for all $i \neq j$. By the compactness of K, (x_n) has a convergent subsequence (x_{n_i}). However, we still have $d(x_{n_i}, x_{n_j}) \geqslant \varepsilon$ for all $i \neq j$, so this is impossible. $\Diamond$

Exercise 3.10 Show that if a Cauchy sequence has a convergent subsequence, then the whole sequence is convergent (to the same limit). Deduce that if K is a compact subset of a metric space (X, d), then the metric space (K, d) is complete.

Theorem 3.9 solves the problem of generalizing Theorem 3.6, but at the expense of introducing the concept of total boundedness, which is much less straightforward than boundedness. However, we shall be interested only in compact subsets of $\mathbb{R}^n$, so we shall take advantage of the simpler description of compact sets in these spaces. The idea of total boundedness will resurface in the chapter on Kolmogorov dimension.

Sups and infs

Definition 3.11 Given a set $E \subseteq \mathbb{R}$, the *supremum* of E, denoted $\sup E$, is the least $x \in \mathbb{R}$ (if such exists) such that $x \geqslant y$ for every $y \in E$. It is also called the *least upper bound* of E.

One of the most basic properties of $\mathbb{R}$ is the following.

Proposition 3.12 *Every non-empty subset of $\mathbb{R}$ which is bounded above has a supremum.*

This proposition can be an axiom in some formulations of analysis. Notice that it fails if $\mathbb{R}$ is replaced by $\mathbb{Q}$ because $\{x \in \mathbb{Q} : x^2 < 2\}$ is a bounded non-empty set with no supremum in $\mathbb{Q}$.

Definition 3.13 Given a set $E \subseteq \mathbb{R}$, the *infimum* of E, denoted $\inf E$, is the greatest $x \in \mathbb{R}$ (if such exists) such that $x \geqslant y$ for every $y \in E$. It is also called the *greatest lower bound* of E.

Naturally, we have a result corresponding to Proposition 3.12: every non-empty subset of $\mathbb{R}$ which is bounded below has an infimum.

Examples 3.14

$$\sup(0,1) = 1, \qquad \inf(0,1) = 0,$$
$$\sup[0,1] = 1, \qquad \inf[0,1] = 0.$$

When $\sup E \in E$ then $\sup E$ is the *maximum* of E, denoted $\max E$ and if $\inf E \in E$ then $\inf E = \min E$, the *minimum* of E. Thus

$$\max[0,1] = 1, \qquad \min[0,1] = 0,$$

but $\max(0,1)$ and $\min(0,1)$ do not exist. The point about sups and infs is that they exist for *all* non-empty bounded subsets of $\mathbb{R}$. Indeed, if one is prepared to define $\sup E = +\infty$ when E is not bounded above and $\inf E = -\infty$ when E is not bounded below, then sups and infs exist for all non-empty subsets of $\mathbb{R}$. Fortunately, compactness often allows us to prove the existence of maxima and minima.

Proposition 3.15 *Non-empty compact subsets of $\mathbb{R}$ have maxima and minima.*

Proof. If K is a non-empty compact subset of $\mathbb{R}$, then K is bounded and so $\sup K$ exists, finite; say $\sup K = u$. For each $n \in \mathbb{N}$, $u - 1/n$ is not an upper bound for K. Therefore there exists $x_n \in K$ with

$$u - 1/n < x_n \leqslant u. \tag{1}$$

Since K is compact, the sequence (x_n) has a subsequence $x_{n_i} \to x \in K$. However, by (1), we have $x_{n_i} \to u$. Therefore $u = x \in K$. The proof for minima is similar. $\Diamond$

Proposition 3.16 *If $f : X \to \mathbb{R}$ is a continuous function on a metric space X and $\emptyset \neq K \subseteq X$ is compact, then f is bounded on K and attains its bounds: i.e. there exist $u, v \in K$ such that $f(u) = \inf_{x \in K} f(x) > -\infty$ and $f(v) = \sup_{x \in K} f(x) < \infty$.*

Proof. Since f is continuous and K is compact, it follows from Theorem 3.3(3) that $f(K)$ is a (non-empty) compact subset of $\mathbb{R}$. Therefore $\max f(K)$ and $\min f(K)$ exist, which is the desired result. $\Diamond$

4 Countability (mainly revision)

The notion of countability is very important in functional analysis; for example, it plays a key rôle in Baire's Category Theorem from which several basic theorems are deduced.

Definition 4.1 A set C is said to be **countable** if either it is empty or there is a surjection $\theta : \mathbb{N} \to C$, (where $\mathbb{N} := \{1, 2, 3, \ldots\}$). A set is said to be **uncountable** if it is not countable.

In other words, C is countable if it may be written

$$C = \{c_1, c_2, c_3, \ldots\},$$

possibly with repetitions. (This rewriting of the definition results from writing c_i for $\theta(i)$.)

Let us remove the repetitions, i.e. define $\phi : \mathbb{N} \to C$ by letting $\phi(n) = \theta(i)$ with i minimal such that

$$\theta(i) \notin \{\phi(1), \ldots, \phi(n-1)\}.$$

Then either ϕ is a bijection between $\mathbb{N}$ and C or the definition of $\phi(n)$ fails at some point; in which case, ϕ is a bijection between $\{1, 2, \ldots, n-1\}$ and C. In the former case, we say that C is **countably infinite**; in the latter, (which includes the case $C = \emptyset$) that C is **finite**. In both cases, the inverse of ϕ gives us an injection of C into $\mathbb{N}$. (In the case $C = \emptyset$, the mapping ϕ is the empty mapping.) To summarise:

Proposition 4.2 (new) *For a set C, the following are equivalent:*

(i) C is countable; i.e. $C = \emptyset$ or there is a surjection $\mathbb{N} \to C$;

(ii) there is an injection $C \to \mathbb{N}$;

(iii) there is a bijection between C and either $\mathbb{N}$ or the set $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N} \cup \{0\}$.

Warning: we say that a set C is **countably infinite** if there is a bijection between C and $\mathbb{N}$. In PMA344 the word 'countable' was used for this. Our definition of 'countable' is 'countably infinite or finite'.

Examples 4.3 (i) The integers are countable:

$$\mathbb{Z} = \{0, +1, -1, +2, -2, +3, \ldots\}.$$

(ii) The positive rationals are countable:

$$\mathbb{Q}^+ = \left\{\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{1}, \ldots\right\},$$

with repetitions. Note that we have grouped numbers with the sum numerator + denominator equal to 2, then 3, 4, 5, *et cetera*.

(iii) The fact that the set $\mathbb{Q}$ of all rationals is countable is easily proved by combining the two previous ideas.

One of the most useful results for proving sets countable is the following.

Theorem 4.4 *Every countable union of countable sets is countable. That is, if each of the set A_i is countable ($i = 1, 2, 3, \ldots$), then $A = \bigcup_{i=1}^{\infty} A_i$ is countable.*

Proof. Clearly, we may assume that the A_i are all non-empty. Let

$$\begin{array}{rcl} A_1 & = & \{a_{11}, a_{12}, a_{13}, a_{14}, \ldots\}; \\ A_2 & = & \{a_{21}, a_{22}, a_{23}, a_{24}, \ldots\}; \\ A_3 & = & \{a_{31}, a_{32}, a_{33}, a_{34}, \ldots\}; \\ A_4 & = & \{a_{41}, a_{42}, a_{43}, a_{44}, \ldots\}; \\ & \ldots & \end{array}$$

(Note that the case of finitely many A_i is included by the simple expedient of repeating the A_i's in the enumeration.) Then

$$A = \{a_{11}, a_{21}, a_{12}, a_{13}, a_{22}, a_{31}, a_{41}, a_{32}, a_{23}, a_{14}, \ldots\},$$

(with possible repetitions). $\Diamond$

The other basic ways of inferring countability of some sets from the countability of others are contained in the following easy proposition.

Proposition 4.5 (new) *(i) If A is countable and $f : A \to B$ is a surjection, then B is countable.*

(ii) If A is countable and $g : B \to A$ is an injection, then B is countable. In particular, subsets of countable sets are countable.

Proof. For (i): Assume A is countable. If $A = \emptyset$ then $B = \emptyset$. Otherwise, there is a surjection $\theta : \mathbb{N} \to A$ so we have a surjection $f\theta : \mathbb{N} \to B$, showing that B is countable.

For (ii): if A is countable then, by Proposition 4.2, there is an injection $\phi : A \to \mathbb{N}$; hence we have an injection $\phi g : B \to \mathbb{N}$, which proves the countability of B, by Proposition 4.2 again. $\Diamond$

The key fact that makes countability interesting is that the reals are uncountable.

Theorem 4.6 *Each of the sets $[0, 1)$, $\mathbb{R}$ and $\mathbb{R} \setminus \mathbb{Q}$ (the irrationals) is uncountable.*

Proof. We prove first that $[0, 1)$ is uncountable, and the rest will follow easily. Every $x \in [0, 1)$ has a decimal expansion $x = 0.x_1x_2x_3\ldots$ not ending in an infinite string of nines. Suppose $[0, 1)$ is countable. Clearly $[0, 1)$ is infinite; let

$$[0, 1) = \{x^{(1)}, x^{(2)}, x^{(3)}, \ldots\}$$

where

$$\begin{array}{rcl} x^{(1)} & = & 0.x_1^{(1)}x_2^{(1)}x_3^{(1)}\ldots, \\ x^{(2)} & = & 0.x_1^{(2)}x_2^{(2)}x_3^{(2)}\ldots, \\ x^{(3)} & = & 0.x_1^{(3)}x_2^{(3)}x_3^{(3)}\ldots, \\ & \ldots & \end{array}$$

We now construct a number $y = 0.y_1y_2y_3\ldots$ different from the above by defining

$$y_i = \begin{cases} 0 & \text{if } x_i^{(i)} \neq 0 \\ 1 & \text{if } x_i^{(i)} = 0. \end{cases}$$

Then $0.y_1y_2y_3\ldots$ is the decimal expansion of a number $y \in [0, 1)$ in a form not ending in an infinite string of nines (since it has no nines at all!). Further, there is no N such that $y = x^{(n)}$ because $y_n \neq x_n^{(n)}$, by construction. This contradicts the supposition that $[0, 1) = \{x^{(1)}, x^{(2)}, x^{(3)}, \ldots\}$ and proves the theorem.

The fact that $\mathbb{R}$ is uncountable follows from Proposition 4.5(ii) since if $\mathbb{R}$ were countable, then $[0, 1)$, being a subset of $\mathbb{R}$, would be countable too. Likewise, $[0, 1]$ is uncountable. Finally, we know that $\mathbb{Q}$ is countable, so if $\mathbb{R} \setminus \mathbb{Q}$ were countable, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would be countable, by Theorem 4.4. This is not so; therefore $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. $\Diamond$

Exercise 4.7 A real number is said to be **algebraic** if it is a root of an equation

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0,$$

with integer coefficients $a_n, \ldots, a_0$, (not all zero). By considering the size of the set A_N of all numbers x satisfying such an equation with $|a_n| + \ldots + |a_0| \leqslant N$ and $n \leqslant N$, or otherwise, show that the set of all algebraic numbers is countable.

A real number is **transcendental** if it is not algebraic. Show that the set of all transcendental numbers is uncountable. Deduce that transcendental numbers exist! (Actually, this is the easiest way of proving the existence of transcendental numbers. It is much harder to produce a specific transcendental number and very much harder to prove that interesting numbers such as e and π are transcendental.)

5 Linear Algebra (mainly revision)

We recall some basic definitions and theorems from PMA211:
Definition. A **vector space** V over a field $\mathbb{F}$ is a set with an addition and a scalar multiplication (so that, for $v, w \in V$ and $\alpha \in \mathbb{F}$, there are defined $v + w \in V$, $\alpha v \in V$) satisfying the following axioms:

1. V is an abelian group under addition;
2. $\alpha(v + w) = \alpha v + \alpha w$ for all $v, w \in V$, $\alpha \in \mathbb{F}$;
3. $(\alpha + \beta)v = \alpha v + \beta v$ for all $v \in V$, $\alpha, \beta \in \mathbb{F}$;
4. $\alpha(\beta v) = (\alpha\beta)v$ for all $v \in V$, $\alpha, \beta \in \mathbb{F}$;
5. $1v = v$ for all $v \in V$.

Theorem 5.1 (Subspaces) *Let V be a vector space over the scalar field $\mathbb{F}$ and let W be a subset of V. Then W is a subspace of V (i.e. W with the addition and scalar multiplication operations inherited from V is a vector space) if and only if the following hold:*

1. *$W \neq \emptyset$ (equivalently $0_V \in W$);*
2. *W is closed under addition, i.e. $w_1 + w_2 \in W$ whenever $w_1 \in W$ and $w_2 \in W$;*
3. *W is closed under scalar multiplication, i.e. $\alpha w \in W$ whenever $w \in W$ and $\alpha \in \mathbb{F}$.*

6 Normed spaces

Normed spaces were introduced by the Polish mathematician Stefan Banach in the 1920's, though several other mathematicians were working along similar lines in the same period. Banach's little book "Théorie des opérations linéaires" is justly regarded as the beginning of the subject. It is still worth reading; an English translation is available.
Definition. Let E be a vector space over $\mathbb{C}$. A **norm** on E is a function $\|.\|$ (i.e. $x \mapsto \|x\|$) from E into $\mathbb{R}^+$ $(= \{\lambda \in \mathbb{R} : \lambda \geqslant 0\})$ such that, for $x, y \in E$, $\lambda \in \mathbb{C}$:

(i) $\|x + y\| \leqslant \|x\| + \|y\|$;

(ii) $\|\lambda x\| = |\lambda|\,\|x\|$;

(iii) $\|x\| = 0$ if and only if $x = 0$.

A **normed space** (or 'normed vector space') is a pair $(E, \|.\|)$ where $\|.\|$ is a norm on the vector space E. (We often refer to just 'the normed space E' if there is no ambiguity.)

Remark 6.1 What we have defined here is a *complex* normed space; one could replace $\mathbb{C}$ by $\mathbb{R}$ in this definition to define the notion of a **real normed space**. Many authors combine both theories by saying that the base field is $\mathbb{F}$, say, which could be either $\mathbb{R}$ or $\mathbb{C}$. We shall assume that the base field is $\mathbb{C}$, which is the more useful case (really!), but just note that the real theory is the same except where otherwise noted.

Proposition 6.2 *If $(E, \|.\|)$ is a normed space then*

$$d(x, y) := \|x - y\| \qquad (x, y \in E)$$

defines a metric on E.

Proof. The three axioms for a metric are satisfied since:

1. $\|y - x\| = \|(-1)(x - y)\| = |-1|\,\|x - y\| = \|x - y\|$ $(x, y \in E)$;
2. $\|x - z\| = \|(x - y) + (y - z)\| \leqslant \|x - y\| + \|y - z\|$ $(x, y, z \in E)$;
3. $\|x - y\| = 0 \iff (x - y) = 0 \iff x = y$ $(x, y \in E)$.

$\Diamond$

The metric d is called the 'associated metric', but neither this term nor the symbol d are much used: it will simply be assumed that if we are working in a normed space terms such as 'convergent sequence', 'open set', 'closed set', 'continuous function', are to be understood in the sense of this metric. For example, $x_n \to x$ in E if and only if $\|x_n - x\| \to 0$; and a function $f : E \to \mathbb{C}$ is continuous if and only if for every $x \in E$ and $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\|y - x\| < \delta$ we have $|f(y) - f(x)| < \varepsilon$.

Examples 6.3 (i) The scalar field $\mathbb{C}$ is a normed space; ($\mathbb{R}$ is a real normed space, as is $\mathbb{C}$).

(ii) For $n \in N$, the space $\mathbb{C}^n$, (n-dimensional Euclidean space), with the norm

$$\|(x_1, x_2, \ldots, x_n)\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

(iii) For $n \in N$, the space $\mathbb{C}^n$ with the norm

$$\|(x_1, x_2, \ldots, x_n)\|_1 := \sum_{i=1}^{n} |x_i|.$$

(iv) For $n \in N$, and $1 \leqslant p < \infty$, the space $\mathbb{C}^n$ with the norm

$$\|(x_1, x_2, \ldots, x_n)\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

This includes the two previous examples as the special cases $p = 1, 2$.

(v) For $n \in N$, the space $\mathbb{C}^n$ with the norm

$$\|(x_1, x_2, \ldots, x_n)\|_\infty := \max_{1 \leqslant i \leqslant n} |x_i|.$$

It can be shown that $\|x\|_p \to \|x\|_\infty$ as $p \to \infty$, for all $x \in \mathbb{C}^n$.

(vi) Let T be any set and let $B(T)$ be the set of all bounded complex-valued functions on T:

$$B(T) := \{f : T \to \mathbb{C} \ : \ \sup_{t \in T} |f(t)| < \infty\}.$$

Then $B(T)$ is a vector space which becomes a normed space on defining

$$\|f\|_\infty := \sup_{t \in T} |f(t)| \qquad (f \in B(T)).$$

(vii) The space $C[0,1]$ of all continuous, complex-valued functions on $[0,1]$, with norm

$$\|f\|_\infty := \sup_{0\leqslant t\leqslant 1} |f(t)| \qquad (f \in C[0,1]).$$

(viii) For $1 \leqslant p < \infty$, the space $C[0,1]$ of all continuous, complex-valued functions on $[0,1]$, with norm

$$\|f\|_p := \left(\int_0^1 |f(t)|\, dt\right)^{1/p} \qquad (f \in C[0,1]).$$

(ix) The space $M_n(\mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$, with the 'operator norm'

$$\|A\| := \sup\left\{\frac{\|Ax\|_2}{\|x\|_2 \ : \ x \in \mathbb{C}^n \setminus \{0\}}\right\} \qquad (A \in M_n(\mathbb{C}))\,.$$

Theorem 6.4 (Continuity of algebraic operations) *Let E be a normed space. Then*

1. *the map $x \mapsto \|x\| : E \to \mathbb{R}^+$ is continuous;*
2. *if $x_n \to x$ and $y_n \to y$ in E, then $x_n + y_n \to x + y$;*
3. *if $x_n \to x$ in E and $\lambda_n \to \lambda$ in $\mathbb{C}$, then $\lambda_n x_n \to \lambda x$.*

Proof.

1. We observe that
$$\|x\| = \|(x-y)+y\| \leqslant \|x-y\| + \|y\|$$
so
$$\|x\| - \|y\| \leqslant \|x-y\|. \tag{2}$$
Therefore
$$\|y\| - \|x\| \geqslant -\|x-y\| = -\|y-x\|. \tag{3}$$
Interchanging the rôles of x and y in (3):
$$\|x\| - \|y\| \geqslant -\|x-y\|. \tag{4}$$
Combining (3) and (4):
$$-\|x-y\| \leqslant \|x\| - \|y\| \leqslant \|x-y\|,$$
that is,
$$|\,\|x\| - \|y\|\,| \leqslant \|x-y\|.$$
Therefore $\|.\| : E \to \mathbb{C}$ is continuous: just take $\delta = \varepsilon$ in the characterization of continuity given above. [For students of the Fractals course: we have shown that $\|.\|$ is a Lipschitz function with Lipschitz constant at most 1 (actually it is equal to 1).]

2. If $x_n \to x$ and $y_n \to y$, then $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$, so
$$\begin{aligned}\|(x_n+y_n)-(x+y)\| &= \|(x_n-x)+(y_n-y)\| \\ &\leqslant \|x_n-x\| + \|y_n-y\| \\ \to 0+0=0,\end{aligned}$$
as $n \to \infty$. Therefore $x_n + y_n \to x + y$.

3. If $x_n \to x$ in E and $\lambda_n \to \lambda$ in $\mathbb{C}$, then the sequence (λ_n), being a convergent sequence of complex numbers, must be bounded; say $|\lambda_n| \leqslant L$ for all n. Then

$$\begin{aligned} 0 &\leqslant \|\lambda_n x_n - \lambda x\| \\ &= \|\lambda_n(x_n - x) + (\lambda_n - \lambda)x\| \\ &\leqslant |\lambda_n|\,\|x_n - x\| + |\lambda_n - \lambda|\,\|x\| \\ &\leqslant L\|x_n - x\| + |\lambda_n - \lambda|\,\|x\| \\ &\to L.0 + 0.\|x\| \\ &= 0, \end{aligned}$$

as $n \to \infty$. Therefore $\lambda_n x_n \to \lambda x$.

$\Diamond$

Corollary 6.5 *Let E be a normed space and F a linear subspace of E. Then the closure $\overline{F}$ of F is also a linear subspace.*

Proof.

1. We have $\overline{F} \supseteq F \neq \emptyset$, so $\overline{F} \neq \emptyset$.
2. If $x, y \in \overline{F}$ then there exist sequences $(x_n), (y_n) \subseteq F$ with $x_n \to x$ and $y_n \to y$. Since F is a subspace, $x_n + y_n \in F$ for all n. But $x_n + y_n \to x + y$, so $x + y \in \overline{F}$.
3. If $x \in \overline{F}$ and $\lambda \in \mathbb{C}$, then there exists a sequence $(x_n) \subseteq F$ with $x_n \to x$. Since F is a subspace, $\lambda x_n \in F$ for all n. But $\lambda x_n \to \lambda x$ (a special case of the previous proposition, with a constant sequence of scalars), so $\lambda x \in \overline{F}$.

$\Diamond$

Example 6.6 Let T be any set and let $B(T)$ be the set of all bounded complex-valued functions on T with

$$\|f\| := \sup_{t \in T} |f(t)| \qquad (f \in B(T)).$$

We prove that $\|.\|$ is a norm, and, in proving properties (i) and (ii), we incidentally prove that $B(T)$ is closed under addition and scalar multiplication, which is a large part of demonstrating that it is a vector space.

(i) If $t \in T$ and $f, g \in B(T)$, then

$$|(f+g)(t)| = |f(t) + g(t)| \leqslant |f(t)| + |g(t)| \leqslant \|f\| + \|g\| < \infty.$$

Therefore

$$\sup_{t \in T} |(f+g)(t)| \leqslant \|f\| + \|g\| < \infty,$$

so $f + g \in B(T)$ and $\|f + g\| \leqslant \|f\| + \|g\|$.

(ii) If $\lambda \in \mathbb{C}$, and $f \in B(T)$, then

$$\sup_{t \in T} |(\lambda f)(t)| = \sup_{t \in T}(|\lambda||f(t)|) = |\lambda| \sup_{t \in T} |f(t)| = |\lambda|\,\|f\| < \infty.$$

Therefore $\lambda f \in B(T)$ and $\|\lambda f\| = |\lambda|\,\|f\|$.

(iii)

$$\begin{aligned} \|f\| = 0 \quad &\iff \quad \sup_{t \in T} |f(t)| = 0 \\ &\iff \quad f(t) = 0 \text{ for all } t \in T \\ &\iff \quad f = 0. \end{aligned}$$

We could likewise consider the real normed space $B_{\mathbb{R}}(T)$ of bounded real-valued functions on T.

Example 6.7 A special case of $B(T)$ is when $T = \mathbb{N}(= \{1, 2, 3, \ldots\})$. Then $B(T)$ is the space ℓ^∞ of all bounded (complex) sequences. We naturally write x_n instead of $f(t)$, so

$$\ell^\infty = \{x = (x_n) : \sup_n |x_n| < \infty\},$$

and we write the norm as

$$\|x\|_\infty = \sup_n |x_n|.$$

Exercise 6.8 Show that ℓ^∞ is non-separable. (Hint: let

$$U = \{x \in \ell^\infty : x_n \in \{0, 1\}(n \in \mathbb{N})\}.$$

Prove that U is uncountable (this can be deduced from the uncountability of $[0, 1]$ or proved directly in the same way as for $[0, 1]$). Prove that for any two distinct points $u, v \in U$, we have $B(u; \frac{1}{2}) \cap B(v; \frac{1}{2}) = \emptyset$.)

Exercise 6.9 Let c_{00} denote the linear subspace of ℓ^∞ consisting of all eventually zero sequences:

$$c_{00} = \{x = (x_n) : \exists N = N(x),\ x_n = 0 \quad (n \geqslant N)\}.$$

Check that c_{00} is a linear subspace and that its closure is the space c_0 of all sequences convergent to zero:

$$c_0 = \{x = (x_n) :\ x_n \to 0 \text{ as } n \to \infty\}.$$

Definition 6.10 A **Banach space** is a normed space which is complete in the associated metric.

Let us write this out in terms of the norm. A sequence (x_n) in a normed space E is a **Cauchy sequence** if, for all $\varepsilon > 0$ [however small], there exists a number N [which will probably be very large if ε is very small] such that for all $p, q > N$ we have $\|x_p - x_q\| < \varepsilon$. (The bits in square brackets are psychology: remove them to get the formal definition!) A normed space E is a Banach space if and only if every Cauchy sequence in E converges to some point of E.

Example 6.11 The space $(c_{00}, \|.\|_\infty)$ is incomplete, since the sequence $(x^{(n)})$ given by

$$x^{(n)} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots\right)$$

is Cauchy, but not convergent in c_{00}. It converges in c_0, to the point

$$x = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right),$$

but this point is not in c_{00}. If it were to converge in c_{00}, then it would have two different limits in c_0, which is impossible.

All the other normed spaces mentioned so far are Banach spaces — though this remains to be proved.

Definition 6.12 Let E be a normed space. Let $\sum_{n=1}^\infty x_n$ be a **series** in E. This means that all the x_n are elements of E and the series is said to be convergent with sum s if and only if the sequence

$$s_n := \sum_{k=1}^n x_k = x_1 + x_2 + \ldots + x_n \qquad (n = 1, 2, 3, \ldots)$$

converges to s as $n \to \infty$. We say that the series is **absolutely convergent** if the series (of real numbers)

$$\sum_{n=1}^\infty \|x_n\|$$

is convergent.

Theorem 6.13 *If E is a Banach space, then every absolutely convergent series in E is convergent.*

Proof. Let $\sum_{n=1}^{\infty} x_n$ be absolutely convergent. Let $t_n := \sum_{k=1}^{n} \|x_k\|$. Now (t_n) is a convergent sequence in $\mathbb{R}$, and so it is Cauchy; that is, given $\varepsilon > 0$, we can find N such that

$$|t_p - t_q| < \varepsilon \qquad (p, q \geqslant N).$$

Let

$$s_n := \sum_{k=1}^{n} x_k = x_1 + x_2 + \ldots + x_n \qquad (n = 1, 2, 3, \ldots).$$

Then for all $\varepsilon > 0$, we can find N such that for all $q > p \geqslant N$,

$$\|s_q - s_p\| = \left\| \sum_{k=p+1}^{q} x_k \right\| \leqslant \sum_{k=p+1}^{q} \|x_k\| = t_q - t_p < \varepsilon.$$

Therefore the sequence (s_n) is a Cauchy sequence in E. Since E is complete, the sequence (s_n) converges: i.e., the series $\sum_{n=1}^{\infty} x_n$ is convergent. $\Diamond$

Exercise 6.14 [hard] If E is a normed space such that every absolutely convergent series in E is convergent, then E is a Banach space.

Definition 6.15 The **closed unit ball** E_1 of a normed E is the set

$$E_1 := \{x \in E : \|x\| \leqslant 1\}.$$

The **open unit ball** E_1° of a normed E is the set

$$E_1^\circ := \{x \in E : \|x\| < 1\}.$$

In the notation of the underlying metric space,

$$E_1^\circ = B(0; 1), \qquad E_1 = \overline{B}(0; 1).$$

Rather less frequently, we shall need the balls of radius R:

$$E_R^\circ = B(0; R), \qquad E_R = \overline{B}(0; R).$$

A subset B of a normed space E is said to be **bounded** if, the set $\{\|x\| : x \in B\}$ is a bounded set of real numbers; i.e. if and only if $B \subseteq E_R$ for some $R \geqslant 0$.

It is easy to show (exercise) that a set B is bounded if and only if it is bounded in the sense of the underlying metric space; i.e. if and only if $B \subseteq B(a; R)$ for some $a \in E$ and $R > 0$. The notion of total boundedness in the metric space is very different — as we shall see later.

Lemma 6.16 *If (x_n) is a Cauchy sequence in a normed space E, then (x_n) is bounded.*

(Strictly speaking, we should write '$\{x_n : n \in \mathbb{N}\}$ is bounded'.)

Proof. If (x_n) is Cauchy, then we can find [by taking '$\varepsilon = 1$'] a number N such that $\|x_p - x_q\| < 1$ whenever $p, q \geqslant N$. In particular,

$$\|x_p - x_N\| < 1 \qquad (p \geqslant N),$$

so

$$\|x_p\| \leqslant \|x_p - x_N\| + \|x_N\| < 1 + \|x_N\| \qquad (p \geqslant N).$$

Therefore, we have, for all p,

$$\|x_p\| \leqslant \max\{\|x_1\|, \|x_2\|, \ldots, \|x_{N-1}\|, 1 + \|x_N\|\} = R, \text{ say.}$$

So $\{x_n : n \in \mathbb{N}\} \subseteq E_R$. $\Diamond$

Examples of Banach spaces

If you look in any book on functional analysis you will see lots of examples of normed spaces. To some extent you will find that many of them are similar; they tend to fall into 3 classes.

- *Spaces of n-tuples of complex numbers:* here the addition and scalar multiplication are the 'obvious' ones –
  if $\boldsymbol{x} = (x_1, x_2, \dots x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots y_n)$ then

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, x_2 + y_2, \dots x_n + y_n)$$

$$\text{and} \qquad \lambda \boldsymbol{x} = (\lambda x_1, \lambda x_2, \dots \lambda x_n)$$

- *Spaces of sequences of complex numbers:* the addition and scalar multiplication are the 'obvious' ones –
  if $\boldsymbol{x} = (x_1, x_2, x_3, \dots)$ and $\boldsymbol{y} = (y_1, y_2, y_3, \dots)$ then

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3 \dots)$$

$$\text{and} \qquad \lambda \boldsymbol{x} = (\lambda x_1, \lambda x_2, \lambda x_3, \dots)$$

- *Spaces of complex valued functions, on a set, T, say:* again the addition and scalar multiplication are the 'obvious' ones –

$$(x + y)(t) = x(t) + y(t) \qquad (\lambda x)(t) = \lambda(x(t)).$$

In each example we first specify the elements of the vector space E, and then the norm. It is then necessary to check that:

- we have a vector space, this generally just means checking that E is closed under addition and scalar multiplication;
- the formula for the norm makes sense, and is a (finite) number for each vector;
- the three properties of the norm (i), (ii), (iii) are satisfied.

We shall also, where true, want to prove completeness.

After looking at a few of these examples you should realise that most parts of these steps are trivial for most examples. Therefore we shall progressively spell out fewer details as we go through.

Example 6.17 [Example 6.3(iii)] Let $(E, \|.\|) = (\mathbb{C}^n, \|.\|_1)$, that is

$$\|\boldsymbol{x}\| = \sum_{p=1}^{n} |x_p| \qquad (\boldsymbol{x} = (x_1, x_2, \dots x_n) \in \mathbb{C}^n)\,.$$

That E is a vector space is clear. The formula for the norm makes sense, and is a non-negative real number, since it is a *finite* sum of non-negative real numbers.

(i) Using the inequality (for complex numbers), $|\alpha + \beta| \leqslant |\alpha| + |\beta|$,

$$\|\boldsymbol{x} + \boldsymbol{y}\| = \sum_{p=1}^{n} |x_p + y_p| \leqslant \sum_{p=1}^{n} |x_p| + \sum_{p=1}^{n} |y_p| = \|\boldsymbol{x}\| + \|\boldsymbol{y}\|.$$

(ii) This is trivial:

$$\|\lambda \boldsymbol{x}\| = \sum_{p=1}^{n} |\lambda x_p| = |\lambda| \sum_{p=1}^{n} |x_p| = |\lambda|\, \|\boldsymbol{x}\|.$$

(iii) This is clear since the norm is a sum of non-negative real numbers, and so is zero if and only if each of these is zero. So

$$\|\boldsymbol{x}\| = 0 \iff |x_p| = 0 \quad \text{for all } p \iff x_p = 0 \quad \text{for all } p \iff \boldsymbol{x} = 0.$$

Finally, this space is complete. For let $\left(\boldsymbol{x}^{(k)}\right)$ be a Cauchy sequence in E. (We index by k since n has been fixed as the dimension of the space.) So, given $\varepsilon > 0$, there is a K such that $\|\boldsymbol{x}^{(p)} - \boldsymbol{x}^{(q)}\| < \varepsilon$

whenever $p, q \geqslant K$. Fix m with $1 \leqslant m \leqslant n$. Clearly $|x_m^{(p)} - x_m^{(q)}| \leqslant \sum_{i=1}^n |x_i^{(p)} - x_i^{(q)}| = \|\boldsymbol{x}^{(p)} - \boldsymbol{x}^{(q)}\| < \varepsilon$ if $p, q \geqslant K$. So the sequence of complex numbers $\left(x_m^{(k)}\right)$ is Cauchy in $\mathbb{C}$ and so convergent (as $\mathbb{C}$ is complete). So let $x_m = \lim_{k\to\infty} x_m^{(k)}$. Then we get an n-tuple $\boldsymbol{x} = (x_1, x_2, \ldots x_n) \in \mathbb{C}^n$. Then

$$\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\right\| = \sum_{i=1}^n \left|x_i^{(k)} - x_i\right| \to 0,$$

since for each fixed i, we have $|x_i^{(k)} - x_i| \to 0$, and there are a finite number of terms in the sum. That is, $\boldsymbol{x}^{(k)} \to \boldsymbol{x}$ (in this norm). Hence this is a Banach space.

Remark 6.18 We could equally have a *real* normed space $\mathbb{R}^n$ of n-tuples of *real* numbers, with the same formula for the norm. We could give an identical proof that this is a Banach space, just replacing complex numbers by real ones, and using the completeness of $\mathbb{R}$.

Example 6.19 [Example 6.3(v)]
Here $(E, \|.\|) = (\mathbb{C}^n, \|.\|_\infty)$, and

$$\|\boldsymbol{x}\| = \max\{|x_1|, |x_2|, \ldots |x_n|\} \qquad (\boldsymbol{x} = (x_1, x_2, \ldots x_n) \in \mathbb{C}^n)\,.$$

That E is a vector space is clear. The formula for the norm makes sense, and is a non-negative real number, since it the maximum of a *finite* sum of non-negative real numbers.

(i) For any p,

$$|(x+y)_p| = |x_p + y_p| \leqslant |x_p| + |y_p| \leqslant \|\boldsymbol{x}\| + \|\mathbf{y}\|,$$

and therefore, as $\|\boldsymbol{x} + \boldsymbol{y}\| = |(x+y)_p|$ for one such p,

$$\|\boldsymbol{x} + \boldsymbol{y}\| \leqslant \|\boldsymbol{x}\| + \|\boldsymbol{y}\|.$$

(ii) Clearly,

$$\begin{aligned} \|\lambda\boldsymbol{x}\| &= \max\{|\lambda x_1|, |\lambda x_2|, \ldots |\lambda x_n|\} \\ &= |\lambda| \max\{|x_1|, |x_2|, \ldots |x_n|\} = |\lambda|\|\boldsymbol{x}\|. \end{aligned}$$

(iii) Clearly the maximum of a finite set of non-negative real numbers is zero if and only if they are all zero. So

$$\begin{aligned} \|\boldsymbol{x}\| = 0 \quad &\iff \quad |x_p| = 0 \quad \text{for all } p \\ &\iff \quad |x_p| = 0 \quad \text{for all } p \\ &\iff \quad x_p = 0 \quad \text{for all } p \\ &\iff \quad \boldsymbol{x} = 0. \end{aligned}$$

Finally, this space is complete. For let $\left(\boldsymbol{x}^{(k)}\right)$ be a Cauchy sequence in E. So, given $\varepsilon > 0$, there is a K such that $\|\boldsymbol{x}^{(p)} - \boldsymbol{x}^{(q)}\| < \varepsilon$ whenever $p, q \geqslant K$. Fix m with $1 \leqslant m \leqslant n$. Clearly $|x_m^{(p)} - x_m^{(q)}| \leqslant \max_{1\leqslant i\leqslant n} |x_i^{(p)} - x_i^{(q)}| = \|\boldsymbol{x}^{(p)} - \boldsymbol{x}^{(q)}\| < \varepsilon$ if $p, q \geqslant K$. So the sequence of complex numbers $\left(x_m^{(k)}\right)$ is Cauchy in $\mathbb{C}$ and so convergent (as $\mathbb{C}$ is complete). So let $x_m = \lim_{k\to\infty} x_m^{(k)}$. Then we get an n-tuple $\boldsymbol{x} = (x_1, x_2, \ldots x_n) \in \mathbb{C}^n$. Then

$$\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\right\| = \max_{1\leqslant i\leqslant n} \left|x_i^{(k)} - x_i\right| \to 0,$$

since for each fixed i, we have $|x_i^{(k)} - x_i| \to 0$, and there are a finite number of terms in the max. That is, $\boldsymbol{x}^{(k)} \to \boldsymbol{x}$ (in this norm). Hence this is a Banach space.

Example 6.20 [Example 6.3(vi)] This example is a (rather general) space of functions. We take T as any non-empty set. We let $B(T)$ denote the set of all bounded, $\mathbb{C}$-valued functions on T. For $x \in B(T)$ we define

$$\|x\| = \sup\{|x(t)| : t \in T\}.$$

Clearly, by definition of $B(T)$, the definition of the norm makes sense and is a non-negative real number. In fact, we have defined $B(T)$ to be exactly those functions for which the norm is finite. (This is a common technique: defining a 'norm' which could be infinite and then restricting the space to be the set of those elements of finite norm.)

To show that $B(T)$ is a vector space is best done at the same time as the proof of (i) and (ii).
(i) If $t \in T$ and $x, y \in B(T)$ then

$$|(x+y)(t)| = |x(t) + y(t)| \leqslant |x(t)| + |y(t)| \leqslant \|x\| + \|y\|.$$

This shows that $x + y$ is in $B(T)$, since it is bounded (by $\|x\| + \|y\|$), and that

$$\|x + y\| \leqslant \|x\| + \|y\|.$$

(ii) Similarly, for $\lambda \in \mathbb{C}$,

$$\|\lambda x\| = \sup\{|\lambda x(t)| : t \in T\} = |\lambda| \sup\{|x(t)| : t \in T\} = |\lambda|\|x\|.$$

This, in particular, shows that λx is in $B(T)$.

So we now know that $B(T)$ is a vector space, and only have left to prove (iii).
(iii) Since the supremum of a set of non-negative real numbers is zero if and only if each of them is zero it is clear that $\|x\| = 0$ iff $x(t) = 0$ for all $t \in T$, that is iff $x = 0$.

Finally, the associated metric is

$$d(x, y) = \sup\{|x(t) - y(t)| : t \in T\},$$

so we know from the Real Analysis course (Proposition 3.8) that $B(T)$ is complete.

(Actually, the space considered in the Real Analysis course is the space $B_{\mathbb{R}}(T)$ of bounded, real-valued functions on the set T, but the same proof works.)

Example 6.21 Let X be a compact metric space, and let $C(X)$ be the vector space of all continuous complex-valued functions on X. (A simple, paradigmatic, example is the case $X = [a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$. We then write $C[a, b]$ rather than $C([a, b])$.) Recall that because X is a compact, each $f \in C(X)$ is bounded (Proposition 3.16), so that $C(X) \subseteq B(X)$, in the notation of the previous example. We give $C(X)$ the norm inherited from this larger space:

$$\|f\| = \sup\{|t(t)| : t \in X\} \qquad (f \in C(X)).$$

Notice that (Proposition 3.16 again), this norm is a maximum, not just a supremum;

$$\|x\| = \max\{|x(t)| : t \in [a, b]\}.$$

That the norm has properties (i) to (iii) follows since these are true in the bigger space $B(X)$. All else we need to show is that we have a vector space, that is if x and y are continuous, and $\lambda \in \mathbb{R}$ then $x + y$ and λx are continuous. This is again true by elementary theorems in analysis. Again the completeness was verified in the Real Analysis course (Corollary 3.10).

Example 6.22 Let $P[0, 1]$ be the vector space of all polynomials of the form

$$p(t) = a_0 + a_1 t + \ldots + a_n t^n \qquad (t \in [0, 1]),$$

where $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in \mathbb{C}$. So the elements of the space are really *functions* which are given by polynomials. This is a vector subspace of the above space $C[0, 1]$ and we give it the inherited norm:

$$\|p\| = \max\{|p(t)| : t \in [0, 1]\}.$$

Again, there is nothing to prove, as we know the bigger space is a normed vector space, and we have clearly got a vector subspace.

This space is *not* complete. In fact, by Weierstrass's Theorem (Real Analysis Theorem 7.3), $P[0, 1]$ is dense in $C[0, 1]$.

Example 6.23 [(Complex) Inner Product Spaces] Let E be a vector space over $\mathbb{C}$. An **inner product** on E is defined to be a map $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{C}$ satisfying, for each $x, y, z \in E$ and each $\lambda, \mu \in \mathbb{C}$,

$$\begin{aligned}
\langle \lambda x + \mu y, z\rangle &= \lambda\langle x, x\rangle + \mu\langle y, z\rangle \\
\langle y, x\rangle &= \overline{\langle x, y\rangle} \\
\langle x, x\rangle &\geqslant 0 \quad \text{(i.e. real, non-negative)} \\
\langle x, x\rangle = 0 &\Leftrightarrow x = 0.
\end{aligned}$$

We then call E an **inner product space** (IPS).
For $x \in E$ we define

$$\|x\| = \langle x, x\rangle^{1/2}.$$

Proposition 6.24 *If E is an IPS, then $(E, \|\cdot\|)$ is a normed space.*

Definition 6.25 If this normed space is complete we call the IPS E a **Hilbert space**.

Proof. We first observe that

$$\langle x, \lambda y\rangle = \overline{\langle \lambda y, x\rangle} = \overline{\lambda\langle y, x\rangle} = \overline{\lambda}\,\overline{\langle y, x\rangle} = \overline{\lambda}\langle x, y\rangle \quad (x, y \in V, \lambda \in \mathbb{C}). \tag{5}$$

Checking the axioms for a norm in reverse order, we have, first that

$$\|x\| = 0 \iff \langle x, x\rangle^{1/2} = 0 \iff \langle x, x\rangle = 0 \iff x = 0.$$

Then

$$\begin{aligned}
\|\lambda x\|^2 &= \langle \lambda x, \lambda x\rangle \\
&= \lambda\overline{\lambda}\langle x, x\rangle \qquad \text{by (5)} \\
&= |\lambda|^2\langle x, x\rangle \\
&= |\lambda|^2\|x\|^2.
\end{aligned}$$

so that $\|\lambda x\| = |\lambda|\|x\|$.

Finally, we need to check the first axiom: the triangle inequality. For this we need the following key result.

Lemma 6.26 (The Cauchy-Schwarz inequality)

$$|\langle x, y\rangle| \leqslant \|x\|\|y\|.$$

Proof of Lemma. Suppose $x \neq 0$: otherwise the inequality is trivially true since both sides reduce to 0. We define

$$\lambda = \frac{-\overline{\langle x, y\rangle}}{\langle x, x\rangle}.$$

Then

$$\begin{aligned}
0 &\leqslant \langle \lambda x + y, \lambda x + y\rangle \\
&= \langle \lambda x, \lambda x\rangle + \langle y, y\rangle + \langle \lambda x, y\rangle + \langle y, \lambda x\rangle \\
&= \lambda\overline{\lambda}\langle x, x\rangle + \langle y, y\rangle + \lambda\langle x, y\rangle + \overline{\lambda\langle x, y\rangle} \\
&= |\lambda|^2\|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,(\lambda\langle x, y\rangle) \\
&= \|y\|^2 + \frac{|\langle x, y\rangle|^2}{\|x\|^2} - 2\,\mathrm{Re}\,\frac{\langle x, y\rangle\overline{\langle x, y\rangle}}{\|x\|^2} \\
&= \|y\|^2 + \frac{|\langle x, y\rangle|^2}{\|x\|^2} - 2\frac{|\langle x, y\rangle|^2}{\|x\|^2} \\
&= \|y\|^2 - \frac{|\langle x, y\rangle|^2}{\|x\|^2}
\end{aligned}$$

Therefore

$$\frac{|\langle x, y\rangle|^2}{\|x\|^2} \leqslant \|y\|^2.$$

So multiplying by $\|x\|^2$ and taking positive square roots we get

$$|\langle x, y\rangle| \leqslant \|x\|\|y\|.$$

$\Diamond$

Now, if $x, y \in E$ then

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \langle y, x\rangle \\
&= \langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \overline{\langle x, y\rangle} \\
&= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\langle x, y\rangle \\
&\leqslant \|x\|^2 + \|y\|^2 + 2|\langle x, y\rangle| \\
&\leqslant \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \text{ (by the Cauchy–Schwarz inequality)} \\
&= (\|x\| + \|y\|)^2.
\end{aligned}$$

Therefore $\|x+y\| \leqslant \|x\| + \|y\|$, by taking non-negative square roots.

This completes the proof that we do have a norm. $\Diamond$

We now give particular examples of IPS's (and Hilbert spaces).

Example 6.27 [Example 6.3(ii)] Take, $E = \mathbb{C}^n$ and define

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{p=1}^{n} x_p \overline{y_p} \qquad (\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n).$$

This is complete and so is a Hilbert space; observe that the norm is given by

$$\|\boldsymbol{x}\| = \left(\sum_{p=1}^{n} |x_p|^2\right)^{1/2} \qquad (\boldsymbol{x} \in \mathbb{C}^n).$$

This is the ℓ^2 norm $\|.\|_2$. The completeness is proved in much the same as in Example 6.17. It is also a simplified version of the proof given in the next example.

Example 6.28 We let $E = \ell^2(\mathbb{N})$ (or just ℓ^2) denote the set of all sequences $\boldsymbol{x} = (x_n)_{n=1}^{\infty}$ of complex numbers for which the series $\sum_{n=1}^{\infty} |x_n|^2$ converges (i.e. has a finite sum). We put

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{n=1}^{\infty} x_n \overline{y_n} \qquad (\boldsymbol{x}, \boldsymbol{y} \in \ell^2). \tag{6}$$

We claim that this is a Hilbert space. Observe also that the formula for the norm on this space is

$$\|\boldsymbol{x}\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}.$$

Proposition 6.29 *The space ℓ^2 is a Hilbert space.*

Proof. We must first show that $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ makes sense for $\boldsymbol{x}, \boldsymbol{y} \in \ell^2$. Consider the following partial sums:

$$\begin{aligned}
\sum_{n=1}^{N} |x_n \overline{y_n}| &\leqslant \left(\sum_{n=1}^{N} |x_n|^2\right)^{1/2} \left(\sum_{n=1}^{N} |y_n|^2\right)^{1/2}, \text{ by Cauchy-Schwarz,} \\
&\leqslant \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |y_n|^2\right)^{1/2} \\
&= \|\boldsymbol{x}\|\,\|\boldsymbol{y}\|,
\end{aligned}$$

for all N. The series $\sum_{n=1}^{\infty} |x_n \overline{y_n}\,|$ is convergent (because its partial sums are bounded). Thus the series $\sum_{n=1}^{\infty} x_n \overline{y_n}$ is absolutely convergent, and so convergent.

We must next show that ℓ^2 is a vector space over $\mathbb{C}$. Let $\boldsymbol{x}, \boldsymbol{y} \in \ell^2$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} |x_n + y_n|^2 &\leqslant \sum_{n=1}^{\infty} |x_n|^2 + \sum_{n=1}^{\infty} |y_n|^2 + 2\sum_{n=1}^{\infty} |x_n||y_n| \\ &= \sum_{n=1}^{\infty} |x_n|^2 + \sum_{n=1}^{\infty} |y_n|^2 + 2\sum_{n=1}^{\infty} |x_n\overline{y_n}| < \infty,\end{aligned}$$

so that $\boldsymbol{x} + \boldsymbol{y} \in \ell^2$. Also,

$$\sum_{n=1}^{\infty} |\lambda x_n|^2 = |\lambda|^2 \sum_{n=1}^{\infty} |x_n|^2 < \infty,$$

so that $\lambda \boldsymbol{x} \in \ell^2$. Hence ℓ^2 is a vector space over $\mathbb{C}$.

We next show that $\langle .,. \rangle$ is an inner product. Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \ell^2$ and $\lambda, \mu \in \mathbb{C}$. Then it is clear, by inspection, that:

$$\begin{aligned}\langle \lambda \boldsymbol{x} + \mu \boldsymbol{y}, \boldsymbol{z} \rangle &= \lambda \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \mu \langle \boldsymbol{y}, \boldsymbol{z} \rangle; \\ \langle \boldsymbol{y}, \boldsymbol{x} \rangle &= \overline{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}; \\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle &= \sum_{n=1}^{\infty} x_n \overline{x_n} = \sum_{n=1}^{\infty} |x_n|^2 \geqslant 0.\end{aligned}$$

And hence also that

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \Leftrightarrow |x_n|^2 = 0 \text{ for all } n \Leftrightarrow \mathbf{x} = \mathbf{0}.$$

Finally, we show that ℓ^2 is complete. Let $(\boldsymbol{x}^{(\ell)})$ be a Cauchy sequence in ℓ^2. Then given $\varepsilon > 0$ we can find $L \in \mathbb{N}$ such that

$$\left\| \boldsymbol{x}^{(k)} - \boldsymbol{x}^{(\ell)} \right\| = \left(\sum_{n=1}^{\infty} \left| x_n^{(k)} - x_n^{(\ell)} \right|^2 \right)^{1/2} < \varepsilon^{1/2} \qquad (k, \ell \geqslant L).$$

Therefore

$$\sum_{n=1}^{\infty} \left| x_n^{(k)} - x_n^{(\ell)} \right|^2 < \varepsilon \qquad (k, \ell \geqslant L).$$

Therefore, for all $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \left| x_n^{(k)} - x_n^{(\ell)} \right|^2 < \varepsilon \qquad (k, \ell \geqslant L). \tag{7}$$

The remainder consists of three steps which are standard for proofs of completeness.

(A) Construct $\boldsymbol{x}$. Fixing n we deduce from (7) that

$$\left| x_n^{(k)} - x_n^{(\ell)} \right|^2 < \varepsilon \qquad (k, \ell \geqslant L).$$

i.e.

$$\left| x_n^{(k)} - x_n^{(\ell)} \right| < \varepsilon^{1/2} \qquad (k, \ell \geqslant L).$$

Therefore the sequence of complex numbers $(x_n^{(k)})$ is Cauchy in $\mathbb{C}$, and so converges. So let

$$x_n = \lim_{k \to \infty} x_n^{(k)} \qquad (n \in \mathbb{N}).$$

This gives us a sequence $\boldsymbol{x} = (x_n)$ of complex numbers.

(B) Show that $\boldsymbol{x} \in \ell^2$. To do this we use the fact that the sequence $\left(\boldsymbol{x}^{(k)}\right)$, being Cauchy, is bounded, say $\left\|\boldsymbol{x}^{(k)}\right\| \leqslant M$ for all k. So

$$\left(\sum_{n=1}^{\infty} \left|x_n^{(k)}\right|^2\right)^{1/2} \leqslant M \qquad \text{for all } k,$$

i.e.

$$\sum_{n=1}^{\infty} \left|x_n^{(k)}\right|^2 \leqslant M^2 \qquad \text{for all } k.$$

Therefore

$$\sum_{n=1}^{N} \left|x_n^{(k)}\right|^2 \leqslant M^2 \qquad \text{for all } k, N.$$

Therefore, letting $k \to \infty$,

$$\sum_{n=1}^{N} |x_n|^2 \leqslant M^2 \qquad \text{for all } N,$$

and letting $N \to \infty$, we get

$$\sum_{n=1}^{\infty} |x_n|^2 \leqslant M^2 < \infty.$$

So we have shown that $\boldsymbol{x} \in \ell^2$.

(C) Show that $\left(\boldsymbol{x}^{(k)}\right)$ converges to $\boldsymbol{x}$. We return to (7), fix N, fix $k \geqslant L$, and let $\ell \to \infty$. Then

$$\sum_{n=1}^{N} \left|x_n^{(k)} - x_n\right|^2 = \lim_{\ell \to \infty} \sum_{n=1}^{N} \left|x_n^{(k)} - x_n^{(\ell)}\right|^2 \leqslant \varepsilon \qquad (N \in \mathbb{N},\ k \geqslant L).$$

Letting $N \to \infty$ we get

$$\sum_{n=1}^{\infty} \left|x_n^{(k)} - x_n\right|^2 \leqslant \varepsilon \qquad (k \geqslant L).$$

So

$$\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\right\| \leqslant \varepsilon^{1/2} \qquad (k \geqslant L).$$

Therefore, since ε is arbitrary, $\boldsymbol{x}^{(k)} \to \boldsymbol{x}$. So $\left(\boldsymbol{x}^{(k)}\right)$ converges, and we have shown that ℓ^2 is complete, i.e. is a Hilbert space. $\Diamond$

Example 6.30 We define $\ell^1(\mathbb{N})$, or just ℓ^1, to be the space of all sequences $\boldsymbol{x} = (x_n)_{n=1}^{\infty}$ of complex numbers, for which the series $\sum_{n=1}^{\infty} |x_n|$ converges. We then define

$$\|\boldsymbol{x}\|_1 = \sum_{n=1}^{\infty} |x_n| \,.$$

Then ℓ^1 is a Banach space (exercise).

Example 6.31 Extending the previous two examples, we define $\ell^p(\mathbb{N})$, or just ℓ^p, for $1 \leqslant p < \infty$, to be the space of all sequences $\boldsymbol{x} = (x_n)_{n=1}^{\infty}$ of complex numbers, for which the series $\sum_{n=1}^{\infty} |x_n|^p$ converges. We then define

$$\|\boldsymbol{x}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

It may be shown that ℓ^p is a Banach space. We shall not prove this, but the proof may be found in most textbooks on functional analysis.

Example 6.32 We define the space $\ell^\infty(\mathbb{N})$, or just ℓ^∞, to be the space of all bounded sequences $\boldsymbol{x} = (x_n)_{n=1}^\infty$ of complex numbers, with norm

$$\|\boldsymbol{x}\| = \sup_{n\in\mathbb{N}} |x_n| \,.$$

This space is just $B(T)$ for $T = \mathbb{N}$, in a slightly different notation, so it is a Banach space. It is probably the simplest example of a non-separable Banach space.

Example 6.33 The space

$$c_0 = \{\boldsymbol{x} \in \ell^\infty : x_n \to 0 \text{ as } n \to \infty\}$$

is a closed subspace of ℓ^∞ and is therefore a Banach space in the inherited (sup) norm.

Example 6.34 The space

$$c = \{\boldsymbol{x} \in \ell^\infty : (x_n) \text{ is convergent}\}$$

is also a closed subspace of ℓ^∞ and therefore a Banach space in the inherited (sup) norm. Alternatively, this space may be viewed as the space $C(X)$ of all continuous functions on the compact metric space $X = \mathbb{N} \cup \{\infty\}$ with the metric defined by

$$d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| \qquad (x, y \in X),$$

with the convention $\frac{1}{\infty} = 0$.

Example 6.35 We can take the vector space $C[a, b]$ of continuous complex-valued functions on the interval $[a, b]$, and define

$$\|x\|_1 = \int_a^b |x(t)| \, dt.$$

This space is not complete (PMA344 Example 2.28).

Example 6.36 The vector space $C[a, b]$ of the previous example can be given an inner product by

$$\langle x, y\rangle = \int_a^b x(t)\overline{y(t)} \, dt.$$

This is then an inner product space which is **not** complete, the proof being almost identical to that for $(C[a, b], \|.\|_1)$.

Example 6.37 Let $C_{2\pi}(\mathbb{R})$ denote the set of all continuous complex-valued functions on the real line which are of period 2π (i.e $x(t + 2\pi) = x(t)$ for all $x \in \mathbb{R}$). Observe that for any $u \in \mathbb{R}$ and $x \in C_{2\pi}(\mathbb{R})$,

$$\int_u^{u+2\pi} x(t) \, dt = \int_0^{2\pi} x(t) \, dt,$$

and that if $x \in C_{2\pi}(\mathbb{R})$ then $|x| \in C_{2\pi}(\mathbb{R})$. For $x \in C_{2\pi}(\mathbb{R})$ we define

$$\|x\|_1 = \int_0^{2\pi} |x(t)| \, dt.$$

Then $C_{2\pi}(\mathbb{R})$ becomes an *incomplete* normed space. As usual, one can define a family of (incomplete) norms $\|.\|_p$ $(1 \leqslant p < \infty)$ on the same space:

$$\|x\|_p = \left(\int_0^{2\pi} |x(t)|^p \, dt\right)^{1/p}.$$

Example 6.38 Following on from the previous example, we can define the sup norm on $C_{2\pi}(\mathbb{R})$. Observe that if $x \in C_{2\pi}(\mathbb{R})$ then x is bounded, and attains its bounds on $\mathbb{R}$ – since it does so on $[0, 2\pi]$. So we can define a norm by

$$\|x\|_\infty = \max\{|x(t)| : t \in [0, 2\pi]\} = \max\{|x(t)| : t \in \mathbb{R}\} \qquad (x \in C_{2\pi}(\mathbb{R})).$$

Then we get a Banach space; in fact, this Banach space may be identified with the space $C(\mathbb{T})$, where $\mathbb{T}$ denotes a circle.

All of the above spaces of continuous functions in which integration is used to define the norm fail to be complete. If you have met the Riemann integral, you may wonder if you could use spaces of Riemann integrable functions in place of continuous functions. That would get round the example of a non-convergent Cauchy sequence in PMA344 Example 2.28. Unfortunately the corresponding spaces of Riemann integrable functions still fail to be complete, though the examples of non-convergent Cauchy sequences are much harder to construct. A yet more sophisticated theory of integration – the Lebesgue integral – was developed about 100 years ago, and if one modifies these examples to include all functions (strictly, equivalence classes of functions) which can be Lebesgue integrated – rather than just continuous ones – then these spaces will become Banach spaces.

7 Linear Mappings and Linear Functionals

Let us recall the definition of a linear mapping between two vector spaces.

Definition 7.1 If E, F are vector spaces, a mapping $T : E \to F$ is **linear** if

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y) \qquad (x, y \in E;\ \lambda, \mu \in \mathbb{C}).$$

In the case when $F = \mathbb{C}$, we say that T is a **linear functional**.

When E, F are normed spaces, we can discuss the continuity of T.

Theorem 7.2 *Let E, F be normed spaces and $T : E \to F$ a linear mapping. Then the following are equivalent:*

(i) T is continuous (i.e. $x_n \to x \ \Rightarrow\ T(x_n) \to T(x)$);

(ii) T is continuous at 0 (i.e. $x_n \to 0 \ \Rightarrow\ T(x_n) \to 0$);

(iii) T is bounded on the closed unit ball E_1 (i.e. there exists $K \in \mathbb{R}^+$ such that $\|T(x)\| \leqslant K$ for all $x \in E$ with $\|x\| \leqslant 1$);

(iv) there exists $K \in \mathbb{R}^+$ such that $\|T(x)\| \leqslant K\|x\|$ for all $x \in E \setminus \{0\}$;

(v) there exists $K \in \mathbb{R}^+$ such that $\|T(x)\| \leqslant K\|x\|$ for all $x \in E$.

Further, (iii)–(v) are valid for the same values of K.

Proof.

(i) $\Rightarrow$ (ii) This is obvious, since $T(0) = 0$.

(ii) $\Rightarrow$ (i) Suppose (ii), and let $x_n \to x$ in E. Then $x_n - x \to 0$, so $T(x_n) - T(x) = T(x_n - x) \to 0$, i.e. $T(x_n) \to T(x)$, in F.

(ii) $\Rightarrow$ (iii) Suppose (iii) is false; then for all $n \in \mathbb{N}$, we can find $x_n \in E_1$ with $\|T(x_n)\| \geqslant n$. Then

$$\left\|n^{-1/2}x_n\right\| = n^{-1/2}\|x_n\| \leqslant n^{-1/2},$$

so

$$n^{-1/2}x_n \to 0$$

in E. But

$$\left\|T\left(n^{-1/2}x_n\right)\right\| = \left\|n^{-1/2}T(x_n)\right\| = n^{-1/2}\|T(x_n)\| \geqslant n^{1/2} \to \infty,$$

so

$$T\left(n^{-1/2}x_n\right) \not\to 0,$$

contradicting (ii).

(iii) $\Rightarrow$ (iv) Suppose (iii). Then for all $x \in E \setminus \{0\}$, we have

$$\left\|\frac{1}{\|x\|}x\right\| = \frac{1}{\|x\|}\|x\| = 1,$$

so, by (iii),

$$\left\|T\left(\frac{1}{\|x\|}x\right)\right\| \leqslant K.$$

But

$$T\left(\frac{1}{\|x\|}x\right) = \frac{1}{\|x\|}T(x),$$

so

$$\frac{1}{\|x\|}\|T(x)\| = \left\|T\left(\frac{1}{\|x\|}x\right)\right\| \leqslant K.$$

Hence

$$\|T(x)\| \leqslant K\|x\|,$$

with the same constant K.

(iv) $\Rightarrow$ (v) This is trivial since $\|T(0)\| = \|0\| = 0 \leqslant K\|0\|$.

(v) $\Rightarrow$ (iii) This is trivial: suppose (v); then

$$\|x\| \leqslant 1 \quad \Rightarrow \quad \|T(x)\| \leqslant K\|x\| \leqslant K.$$

(v) $\Rightarrow$ (ii) Suppose (v): then if $x_n \to 0$ we have $\|Tx_n\| \leqslant K\|x_n\| \to 0$, so $Tx_n \to 0$. Thus T is continuous at 0.

$\Diamond$

Definition 7.3 We say that a linear mapping $T : E \to F$ satisfying the above conditions is **bounded**; (so 'bounded' is equivalent to 'continuous'). The set of all bounded linear mappings from E to F is denoted $\mathcal{B}(E, F)$, or $\mathcal{L}(E, F)$. In the case $E = F$, we write $\mathcal{B}(E)$, or $(\mathcal{E})$, and speak of the bounded linear mappings as **bounded (linear) operators** on E. For $T \in \mathcal{B}(E, F)$, we define the **(operator) norm** $\|T\|$ of T to be the smallest K for which (iii)–(v) hold:

$$\|T\| = \min\{K \in \mathbb{R}^+ : \|Tx\| \leqslant K\|x\| \qquad (x \in E)\}.$$

Then

$$\|T\| \leqslant \|T\|\,\|x\| \qquad (x \in E)$$

and

$$\|T\| \leqslant K\|x\| \quad (x \in E) \quad \Rightarrow \quad \|T\| \leqslant K.$$

Remark 7.4 WARNING: a (non-zero) linear mapping cannot be bounded on the whole of E: 'bounded', applied to linear mappings, will always mean 'bounded on E_1'.

Theorem 7.5 *The space $\mathcal{B}(E,F)$ is a normed space and, if $E \neq \{0\}$, it is complete if and only if F is complete.*

Proof. We show that $\mathcal{B}(E,F)$ is a vector space. Since it is a subset of the vector space of all linear mappings of E into F, all we need do is to show that it is a vector subspace. We show that $\|.\|$ is a norm at the same time.

1. Clearly 0, the zero mapping, is continuous, so $\mathcal{B}(E,F) \neq \emptyset$.

2. If $S, T \in \mathcal{B}(E,F)$, then for all $x \in E$,

$$\begin{aligned} \|(S+T)(x)\| &= \|Sx + Tx\| \\ &\leqslant \|Sx\| + \|Tx\| \\ &\leqslant \|S\|\,\|x\| + \|T\|\,\|x\| \\ &\leqslant (\|S\| + \|T\|)\|x\|. \end{aligned}$$

   Thus $S+T$ is bounded (with $K = \|S\| + \|T\|$), so $S+T \in \mathcal{B}(E,F)$ with $\|S+T\| \leqslant \|S\| + \|T\|$.

3. If $T \in \mathcal{B}(E,F)$ and $\lambda \in \mathbb{C}$, then for all $x \in E$,

$$\|(\lambda T)(x)\| = \|\lambda(Tx)\| = |\lambda|\,\|Tx\| \leqslant |\lambda|\,\|T\|\,\|x\|.$$

   Thus λT is bounded (with $K = |\lambda|\,\|T\|$), so $\lambda T \in \mathcal{B}(E,F)$ (completing the proof that $\mathcal{B}(E,F)$ is a vector space) and

$$\|\lambda T\| \leqslant |\lambda|\,\|T\|. \tag{8}$$

   If $\lambda = 0$, we have $\|\lambda T\| = \|0\| = 0 = |\lambda|\,\|T\|$. If $\lambda \neq 0$, we use (8) with $1/\lambda$ in place of λ to obtain

$$|\lambda|\,\|T\| = |\lambda| \left\| \frac{1}{\lambda}(\lambda T) \right\| = |\lambda| \left| \frac{1}{\lambda} \right| \|\lambda T\| = \|\lambda T\|.$$

   Combining this with (8) shows $\|\lambda T\| = |\lambda|\,\|T\|$.

4. Finally,

$$\begin{aligned} \|T\| = 0 \quad &\iff \quad \|Tx\| \leqslant 0\|x\| \qquad (x \in E) \\ &\iff \quad \|Tx\| = 0 \qquad (x \in E) \\ &\iff \quad Tx = 0 \qquad (x \in E) \\ &\iff \quad T = 0. \end{aligned}$$

   This completes the proof that the operator 'norm' is, indeed, a norm. Therefore $\mathcal{B}(E,F)$ is a normed space.

Suppose F is complete; we show that then $\mathcal{B}(E,F)$ is complete. Let (T_n) be a Cauchy sequence in $\mathcal{B}(E,F)$. Then, for any given $\varepsilon > 0$ we can find N such that

$$\|T_p - T_q\| < \varepsilon \qquad (p, q \geqslant N).$$

Therefore

$$\|T_p x - T_q x\| = \|(T_p - T_q)x\| < \varepsilon\|x\| \qquad (p, q \geqslant N;\ x \in E). \tag{9}$$

Now fix $x \in E$. Then, (9) tells us that the sequence $(T_n x)$ is Cauchy in F. Now F is complete, so this sequence converges, say

$$Tx := \lim_{n\to\infty} T_n x.$$

At this stage, we just have a mapping $T : E \to F$. We show that $T \in \mathcal{B}(E,F)$; i.e., we show: (i) T is linear; (ii) T is bounded. Then we shall show that $T_n \to T$.

(i) To show that T is linear: for $x, y \in E$ and $\lambda, \mu \in \mathbb{C}$,

$$\begin{aligned} T(\lambda x + \mu y) &= \lim_{n\to\infty} T_n(\lambda x + \mu y) \\ &= \lim_{n\to\infty} (\lambda T_n x + \mu T_n y) \\ &= \lambda \lim_{n\to\infty} T_n x + \mu \lim_{n\to\infty} T_n y \\ &= \lambda T x + \mu T y. \end{aligned}$$

(ii) Since (T_n) is Cauchy, it is bounded: say $\|T_n\| \leqslant M$ for all n. Then

$$\|T_n x\| \leqslant M\|x\|$$

for all n and all $x \in E$. But $T_n x \to Tx$, so $\|T_n x\| \to \|Tx\|$. Hence

$$\|Tx\| \leqslant M\|x\|$$

for all $x \in E$. Hence T is bounded; i.e. we have proved $T \in \mathcal{B}(E, F)$.

To show that $T_n \to T$ in $\mathcal{B}(E, F)$, we go back to (9), fix $p \geqslant N$ and let $q \to \infty$. Then

$$T_p x - T_q x \to T_p x - Tx, \text{ so } \|T_p x - T_q x\| \to \|T_p x - Tx\|,$$

so

$$\|(T_p - T)x\| = \|T_p x - Tx\| \leqslant \varepsilon\|x\| \qquad (x \in E;\ p \geqslant N).$$

Thus $T_n \to T$ in $\mathcal{B}(E, F)$, and we have proved that $\mathcal{B}(E, F)$ is complete.

The converse, that $\mathcal{B}(E, F)$ complete implies F complete, is left as an exercise. $\Diamond$

Linear Functionals

Definition 7.6 A **linear functional** on a vector space E is simply a linear mapping into the scalar field (in our case $\mathbb{C}$).

By Theorem 7.2, a linear functional f on a normed space E is continuous if and only if there is a $K \in \mathbb{R}^+$ such that

$$|f(x)| \leqslant K\|x\| \qquad (x \in E),$$

in which case, $\|f\|$ is the smallest K for which this is true.

Definition 7.7 We denote the normed space $\mathcal{B}(E, \mathbb{C})$ of all continuous linear functionals on a normed space E by E' and call it the **dual space** of E.

From Theorem 7.5 we deduce

Corollary 7.8 *For every normed space E, the dual space E' is a Banach space.*

There is a very useful characterization of continuity for linear functionals.

Theorem 7.9 *Let E be a normed space. Then a linear functional f on E is continuous if and only if the vector subspace* $\ker f$ *is closed.*

Proof. We have implicitly assumed that $\ker f$ is a vector subspace: this is easy to check. Indeed the inverse image of a vector subspace under a linear mapping is always a vector subspace.

One direction in the theorem is easy: if f is continuous, then $\ker f = f^{-1}(\{0\})$ is closed because it is the inverse image under a continuous mapping of the closed set $\{0\}$.

Now suppose that f is not continuous. Then f is not continuous at 0, so there exists a sequence $x_n \to 0$ with $f(x_n) \not\to 0$. By taking a subsequence if necessary, we may assume that there exists $\varepsilon > 0$ such that $|f(x_n)| \geqslant \varepsilon$ for all n. Let $x \in E \setminus \ker f$: so $f(x) \neq 0$ and let

$$y_n = \frac{1}{f(x_n)}\, x_n, \qquad y = \frac{1}{f(x)}\, x,$$

Then

$$\|y_n\| = \left|\frac{1}{f(x_n)}\right| \|x_n\| \leqslant \frac{1}{\varepsilon}\|x_n\| \to 0$$

so $y - y_n \to y$. But $f(y_n) = 1$ for all n, and $f(y) = 1$, so $f(y - y_n) = 0$, i.e. $y - y_n \in \ker f$ with $y - y_n \to y \notin \ker f$. Thus $\ker f$ is not closed. This completes the proof. $\Diamond$

8 The Axiom of Choice

For the main theorem in the next section, we shall require an axiom normally regarded as rather suspect — not one of the usual axioms ordinary set theory (Zermelo-Fraenkel set theory (ZF)). This new axiom is the Axiom of Choice (AC).

Definition 8.1 Given a family $\{A_i\}_{i\in I}$ of nonempty sets, we say that a function $f : I \to \bigcup_{i\in I} A_i$ is a **choice function** if $f(i) \in A_i$ for all $i \in I$.

(AC) Every family of nonempty sets has a choice function.

Colloquially: given a collection of non-empty sets it is always possible to make a simultaneous choice of one element from each.

Now in many situations, it is not necessary to invoke (AC) to get a choice function. This is particularly the case if the sets y have additional structure. For example: if the sets y are groups, we can simply choose $f(y)$ as the identity element of each y; if the y are sets of positive integers, we can let $f(y)$ be the least element of y. The Axiom of Choice only comes into play when we need to make **arbitrary** choices, for example if the y are arbitrary sets of reals.

The Tale of the Millionaire's Socks illustrates this distinction. There once was a millionaire (actually, an infinitely rich millionaire) who owned infinitely many pairs of shoes and socks. One day, for want of anything better to do, he decided to try to select one of each pair of shoes and one of each pair of socks. He selected the shoes easily enough, by just taking the left shoe in each case, but (working in (ZF)) he was unable to simultaneously select one from each pair of socks! (Exercise for the reader: work this into a convincing fairy tale!)

Furthermore, (AC) is only needed to make **infinitely many** arbitrary choices **simultaneously**. If $I = \{1\}$, so that we only want to make a choice $f(1)$ from a single non-empty set A_1, then the fact that $A_1 \neq \emptyset$ means that $(\exists z)(z \in A_1)$ so $(\exists f : I \to \bigcup_{i\in I} A_i = A_1)(f(1) \in A_1)$. A (finite) induction argument then shows that if $I_n = \{1, 2, \ldots, n\}$ for any finite n, then there is a choice function f_n. The set C_n of choice functions on I_n is nonempty. However, we cannot make the step to $I = \{1, 2, 3, \ldots\}$ by forming the choice function $f(i) = f_i(i)$, since this would involve having first *chosen* $f_n \in C_n$ for each n, making infinitely many arbitrary choices simultaneously.

The Axiom of Choice was first stated in the early years of the twentieth century (Beppo Levi 1902, Erhard Schmidt 1904; Zermelo 1904). In 1938, Kurt Gödel proved the *consistency* of (AC) — i.e. that it is not possible to disprove (AC) within (ZF). In 1963, Paul Cohen proved the *independence* of (AC): it is not possible to prove (AC) within (ZF).

These proofs may be found in books by the original authors:
Kurt Gödel, "The consistency of the continuum hypothesis", (Princeton U. P., Annals of Mathematics Studies, **3**, 1964), 69pp;
Paul J. Cohen, "Set theory and the continuum hypothesis", (Benjamin, 1966), 154pp;
or in any reasonably advanced book on set theory.

We shall use (AC) in the form of **Zorn's Lemma**. To state this, we need some terminology concerning posets.

Definition 8.2 A **poset** (partially ordered set) is a set X with a binary relation $\leqslant$ satisfying the following axioms:

1. $x \leqslant x$ $(x \in X)$ (reflexive);
2. $x \leqslant y$ & $y \leqslant x \;\Rightarrow\; x = y$ $(x, y \in X)$ (anti-symmetric);
3. $x \leqslant y$ & $y \leqslant z \;\Rightarrow\; x \leqslant z$ $(x, y \in X)$ (transitive).

A poset $(X, \leqslant)$ is **totally ordered** if

$$x, y \in X \;\Rightarrow\; x \leqslant y \text{ or } y \leqslant x.$$

A **chain** in a poset is a totally ordered subset. An element x in a poset $(S, \leqslant)$ is an **upper bound** for a subset $T \subseteq S$ if $x \geqslant t$ for all $t \in T$. (It is not required that x be in T.) A **maximal** element of a poset $(S, \leqslant)$ is an element $s \in S$ such that for all $t \in S$, $t \geqslant s \;\Rightarrow\; t = s$.

Theorem 8.3 (Zorn's Lemma) *Every non-empty poset in which every chain has an upper bound has a maximal element.*

Zorn's Lemma follows from (AC), but it is also an *equivalent* of (AC) — that is, it is possible to deduce (AC) from Zorn's Lemma using only the axioms of (ZF). Proofs may be found in my notes on set theory, but are outside the scope of the present course.

There are many statements which are similarly equivalent in (ZF) to (AC). In fact there are 2 books full of them!

H. Rubin & J. Rubin, 'Equivalents of the Axiom of Choice' (North-Holland, 1963).
H. Rubin & J. Rubin, 'Equivalents of the Axiom of Choice II' (North-Holland).

9 The Hahn-Banach Theorem

We have defined the dual space E' of a normed space E to be the set of all continuous linear functionals on E, but, as yet, we have no guarantee that E' is non-trivial (i.e. $E' \neq \{0\}$) when $E \neq \{0\}$). Algebra tells us that there are plenty of linear functionals, but we want continuous ones. The Hahn-Banach Theorem tells us this, and more: it tells us that there are enough continuous linear functionals to 'separate the points' of E; that is, if $x \neq y$ in E, there is an $f \in E'$ with $f(x) \neq f(y)$. The main theorem is an extension theorem: given a continuous linear functional on a subspace, it can be extended to a continuous linear functional on the whole space.

The route to this important theorem is a rather awkward, as it more easily proved in real normed spaces than in complex ones. We shall begin with an algebraic version of the theorem and introduce norms later.

Definition 9.1 Let E be a *real* vector space. A function $p : E \to \mathbb{R}$ is a **sublinear functional** if:

(i) $p(x + y) \leqslant p(x) + p(y)$ $(x, y \in E)$; and

(ii) $p(\lambda x) = \lambda p(x)$ $(x \in E;\ \lambda \geqslant 0)$.

Examples 9.2 (i) Every linear functional is a sublinear functional: not a useful example for our present purposes.

(ii) If $(E, \|.\|)$ is a real normed space and $K \geqslant 0$, then $p(x) = K\|x\|$ is a sublinear functional on E. (Indeed, it suffices for $\|.\|$ to be a *seminorm* — dropping the '$\|x\| = 0 \;\Rightarrow\; x = 0$' axiom.)

Theorem 9.3 (Hahn–Banach for real vector spaces) *Let E be a real vector space and let p be a sublinear functional on E. Let M be a vector subspace of E and let f be a linear functional on M such that*

$$f(x) \leqslant p(x) \qquad (x \in M).$$

Then there is a linear functional $\overline{f}$ on E extending f, with

$$\overline{f}(x) \leqslant p(x) \qquad (x \in E).$$

Proof. When we say that '$\overline{f}$ extends f' we mean simply that $\overline{f}(x) = f(x)$ $(x \in M)$.

Let $\mathcal{F}$ be the set of all pairs (F, ϕ) consisting of a vector subspace F with $M \leqslant F \leqslant E$ and a linear functional ϕ on F extending f, with $\phi(x) \leqslant p(x)$ $(x \in F)$. We partially order $\mathcal{F}$ by the relation

$$(F_1, \phi_1) \leqslant (F_2, \phi_2) \iff F_1 \leqslant F_2 \text{ and } \phi_2 \text{ extends } \phi_1.$$

It is clear that $\mathcal{F}$ is a poset. It satisfies the hypothesis of Zorn's Lemma, for if $\{(F_i, \phi_i)\}_{i \in I}$ is a chain in $\mathcal{F}$, then it is bounded above by $\left(\bigcup_{i \in I} F_i, \psi\right)$ where ψ is the functional defined on $\bigcup_{i \in I} F_i$ by

$$\psi(x) = \phi_i(x) \qquad (x \in F_i).$$

At first sight, this is not a valid definition: what if $x \in F_i \cap F_j$? How then do we define $\psi(x)$? The difficulty is resolved by the fact that we have a chain, so either $(F_i, \phi_i) \leqslant (F_j, \phi_j)$ or $(F_j, \phi_j) \leqslant (F_i, \phi_i)$. Therefore, either ϕ_i is an extension of ϕ_j or *vice versa*. Either way, $\phi_i(x) = \phi_j(x)$ $(x \in F_i \cap F_j)$, so $\psi(x)$ is well-defined.

By Zorn's Lemma, $\mathcal{F}$ has a maximal element (G, ϕ). We claim that $G = E$, so that ϕ is the required extension $\overline{f}$.

Suppose $G \neq E$. We choose a point $z \in E \setminus G$ and obtain an extension

$$(G \oplus \mathbb{R}z, \psi) \geqslant (G, \phi),$$

contradicting maximality.

To produce the extension ψ of ϕ to the subspace $G \oplus \mathbb{R}z$ spanned by G and z, we only need to choose a suitable value for $\psi(z)$ and then define

$$\psi(y + \lambda z) = \phi(y) + \lambda\psi(z) \qquad (y \in G;\ \lambda \in \mathbb{R}).$$

However, we must have, for all $x = y + \lambda z \in G \oplus \mathbb{R}z$,

$$p(x) - \psi(x) \geqslant 0.$$

For $\lambda = 0$, this is automatically satisfied because $\phi \leqslant p$ on G. For $\lambda \neq 0$, we need

$$\begin{aligned} 0 &\leqslant p(y + \lambda z) - \psi(y + \lambda z) \\ &= p(y + \lambda z) - \phi(y) - \lambda\psi(z) \\ &= \begin{cases} \lambda\left\{p\left(\lambda^{-1}y + z\right) - \phi\left(\lambda^{-1}y\right) - \psi(z)\right\} & (\lambda > 0) \\ -\lambda\left\{p\left(-\lambda^{-1}y - z\right) + \phi\left(\lambda^{-1}y\right) + \psi(z)\right\} & (\lambda < 0). \end{cases} \end{aligned}$$

These inequalities must hold for all elements $\lambda^{-1}y \in G$. Thus, replacing $\lambda^{-1}y$ by y,

$$p(y + z) - \phi(y) \geqslant \psi(z)$$

and

$$-p(-y - z) - \phi(y) \leqslant \psi(z)$$

for all $y \in G$. This is achieved if we can choose $\psi(z)$ such that

$$\sup_{y \in G}\left(-p(-y - z) - \phi(y)\right) \leqslant \psi(z) \leqslant \inf_{y \in G}\left(p(y + z) - \phi(y)\right).$$

Thus all we need to do is to show that

$$\sup_{y_1 \in G}\left(-p(-y_1 - z) - \phi(y_1)\right) \leqslant \inf_{y_2 \in G}\left(p(y_2 + z) - \phi(y_2)\right). \tag{10}$$

If $y_1, y_2 \in G$, then

$$\begin{aligned} \phi(y_2) - \phi(y_1) &= \phi(y_2 - y_1) \\ &\leqslant p(y_2 - y_1) \\ &= p\left((y_2 + z) + (-y_1 - z)\right) \\ &\leqslant p(y_2 + z) + p(-y_1 - z). \end{aligned}$$

Therefore

$$-p(-y_1 - z) - \phi(y_1) \leqslant p(y_2 + z) - \phi(y_2) \qquad (y_1, y_2 \in G).$$

Hence (10), and the theorem is proved. $\Diamond$

Theorem 9.4 (Hahn–Banach) *Let $(E, \|.\|)$ be either a (complex) normed space or a real normed space, and let M be a vector subspace of E. Let f be a continuous linear functional on M. Then there is a continuous linear functional $\overline{f}$ on E extending f with $\|\overline{f}\| = \|f\|$.*

Proof. We prove the real and complex cases separately, beginning with the real case.

Given the continuous linear functional f on M, we define

$$p(x) = \|f\| \, \|x\| \qquad (x \in E),$$

where $\|f\|$ is the norm of f as a linear functional <u>on M</u>. We have already observed (Example 9.2(ii)) that this defines a sublinear functional. Then

$$f(x) \leqslant |f(x)| \leqslant \|f\| \, \|x\| = p(x) \qquad (x \in M).$$

Applying Theorem 9.3, we obtain a linear functional $\overline{f}$ on E extending f and with

$$\overline{f}(x) \leqslant p(x) = \|f\| \, \|x\| \qquad (x \in E).$$

Then

$$-\overline{f}(x) = \overline{f}(-x) \leqslant \|f\| \, \|-x\| = \|f\| \, \|x\| \qquad (x \in E).$$

Thus $\overline{f}$ is continuous, with $\|\overline{f}\| \leqslant \|f\|$. However

$$\begin{aligned} \|\overline{f}\| &= \sup\{|\overline{f}(x)| : x \in E\} \\ &\geqslant \sup\{|\overline{f}(x)| : x \in M\} \\ &= \sup\{|f(x)| : x \in M\} \\ &= \|f\|, \end{aligned}$$

so the desired result follows.

Now let us consider the complex case: given a continuous (complex) linear functional f on M, we write

$$f(x) = f_1(x) + i f_2(x) \qquad (x \in M),$$

where $f_1, f_2 : M \to \mathbb{R}$ are continuous real linear functionals on M, considered as a real normed space. Since $|f_1(x)| \leqslant |f(x)|$, we have $\|f_1\| \leqslant \|f\|$. Now f was linear over the complex field so

$$f(ix) = i f(x) \qquad (x \in M).$$

That is,

$$f_1(ix) + i f_2(ix) = i f_1(x) - f_2(x) \qquad (x \in M).$$

Equating real parts:

$$f_2(x) = -f_1(ix) \qquad (x \in M). \tag{11}$$

Now $(E, \|.\|)$ may be viewed as a real normed space, so we may apply the real case proved above to deduce that there is a continuous real linear functional $\overline{f_1}$ on E extending f_1, with $\|\overline{f_1}\| = \|f_1\| \leqslant \|f\|$. We define $\overline{f_2}$ on E by

$$\overline{f_2}(x) = -\overline{f_1}(ix) \qquad (x \in E).$$

Then, by (11), $\overline{f_2}$ is an extension of f_2 to E. Let

$$\overline{f}(x) = \overline{f_1}(x) + i\overline{f_2}(x) \qquad (x \in E).$$

Then not only is $\overline{f}$ a continuous real linear functional on E, but

$$\begin{aligned}\overline{f}(ix) &= \overline{f_1}(ix) + i\overline{f_2}(ix)\\ &= -\overline{f_2}(x) - i\overline{f_1}(-x)\\ &= i\overline{f_1}(x) - \overline{f_2}(x)\\ &= i\overline{f}(x),\end{aligned}$$

so $\overline{f}$ a complex linear functional on E. To find $\|\overline{f}\|$, suppose $x \in E$ and $\overline{f}(x) = re^{i\theta}$ in modulus–argument form: then $\overline{f}(e^{-i\theta}x) = r$ which is real, so $\overline{f_1}(e^{-i\theta}x) = r$. Therefore

$$|\overline{f}(x)| = r = \left|\overline{f_1}\left(e^{-i\theta}x\right)\right| \leqslant \|f\| \left\|e^{-i\theta}x\right\| = \|f\| \, \|x\|,$$

so $\|\overline{f}\| \leqslant \|f\|$. The reverse inequality follows by the same argument as in the real case. $\Diamond$

Corollary 9.5 *Let x_0 be a non-zero element of a normed space E. Then there exists $f \in E'$ with $\|f\| = 1$ and $f(x_0) = \|x_0\|$. In particular, $E' \neq \{0\}$.*

Proof. We apply the Hahn–Banach Theorem with $M = \operatorname{span}(x_0)$. We define f on M by

$$f(\lambda x_0) = \lambda \|x_0\|,$$

so that $f(x_0) = \|x_0\|$ and

$$\|f\| = \sup_{\lambda \neq 0} \frac{|\lambda| \, \|x_0\|}{\|\lambda x_0\|} = 1.$$

The Hahn–Banach Theorem then allows us to extend f to a continuous linear functional on the whole of E with these same properties. $\Diamond$

Corollary 9.6 *Let $x \neq y$ be distinct elements of a normed space E. Then there exists $f \in E'$ $f(x) \neq f(y)$.*

Proof. Apply the previous corollary to $x_0 = x - y$. $\Diamond$

Corollary 9.7 *Let E be a normed space: then for all $x \in E$*

$$\|x\| = \sup\{|f(x)| : f \in E',\ \|f\| \leqslant 1\}.$$

Note the duality between this and the definition of the norm of a linear functional:

$$\|f\| = \sup\{|f(x)| : x \in E,\ \|x\| \leqslant 1\}.$$

Proof. The inequality

$$\|x\| \geqslant \sup\{|f(x)| : f \in E',\ \|f\| \leqslant 1\}$$

holds because $\|f\| \leqslant 1$ implies $|f(x)| \leqslant \|x\|$. Corollary 9.5 implies the reverse inequality, with the sup actually attained. $\Diamond$

10 Dual spaces

In this section we give some examples of dual spaces.

Theorem 10.1 *The dual of c_0 is ℓ^1. Specifically:*

(i) every $\boldsymbol{x} = (x_1, x_2, \ldots) \in \ell^1$ defines a continuous linear functional $f_{\boldsymbol{x}}$ on c_0 by

$$f_{\boldsymbol{x}}(\boldsymbol{y}) = \sum_{i=1}^{\infty} x_i y_i \qquad (\boldsymbol{y} = (y_1, y_2, \ldots) \in c_0); \tag{12}$$

(ii) every continuous linear functional on c_0 is of this form;

(iii) $\|f_{\boldsymbol{x}}\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \ell^1$, (where $\|f_{\boldsymbol{x}}\|$ is the norm of the linear functional and $\|\boldsymbol{x}\|$ is the ℓ^1-norm of $\boldsymbol{x}$).

Proof.

(i) We must first observe that the sum

$$\sum_{i=1}^{\infty} x_i y_i$$

is (absolutely) convergent for every $\boldsymbol{x} \in \ell^1$ and $\boldsymbol{y} \in c_0$, since

$$\sum_{i=1}^{\infty} |x_i y_i| \leqslant \|\boldsymbol{y}\|_\infty \sum_{i=1}^{\infty} |x_i| < \infty.$$

Thus (12) does define $f_{\boldsymbol{x}}(\boldsymbol{y})$. The proof that $f_{\boldsymbol{x}}$ is linear is straightforward and is omitted. (It is common practice to say this of linearity proofs.) We should, however, prove that $f_{\boldsymbol{x}}$ is continuous. This follows from essentially the same inequalities:

$$|f_{\boldsymbol{x}}(\boldsymbol{y})| \leqslant \sum_{i=1}^{\infty} |x_i y_i| \leqslant \|\boldsymbol{y}\|_\infty \sum_{i=1}^{\infty} |x_i| \leqslant \|\boldsymbol{y}\| \, \|\boldsymbol{x}\|_1.$$

Thus $f_{\boldsymbol{x}} \in (c_0)'$, and we also have $\|f_{\boldsymbol{x}}\| \leqslant \|x\|_1$.

(ii) We must show that an arbitrary $f \in (c_0)'$ is of the form $f_{\boldsymbol{x}}$ for some $\boldsymbol{x} \in \ell^1$. Define $\boldsymbol{e}_n \in c_0$ $(n = 1, 2, 3, \ldots)$ by

$$(\boldsymbol{e}_n)_i = \begin{cases} 1 & (i = n) \\ 0 & (i \neq n). \end{cases}$$

Observation: (not part of the proof) if we had $f = f_{\boldsymbol{x}}$, then we should have $f(\boldsymbol{e}_n) = x_n$. Returning to the proof, we *define* a sequence $\boldsymbol{x} = (x_1, x_2, \ldots)$, using the given functional f, by

$$x_n = f(\boldsymbol{e}_n) \qquad (n = 1, 2, 3, \ldots).$$

Then

$$\begin{aligned}
\sum_{i=1}^{N} |x_i| &= \sum_{i=1}^{N} |f(\boldsymbol{e}_i)| \\
&= \sum_{i=1}^{N} f(a_i \boldsymbol{e}_i), \text{ for suitable } a_i \in \mathbb{C} \text{ with } |a_i| = 1, \\
&= f\left(\sum_{i=1}^{N} a_i \boldsymbol{e}_i\right) \\
&\leqslant \|f\| \left\| \sum_{i=1}^{N} a_i \boldsymbol{e}_i \right\| \\
&= \|f\|.
\end{aligned}$$

This hold for all N, so $\sum_{i=1}^{\infty} |x_i|$ converges, so $\boldsymbol{x} \in \ell^1$. Indeed,

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i| \leqslant \|f\|.$$

We have defined $\boldsymbol{x}$ so that $f_{\boldsymbol{x}}(\boldsymbol{y}) = f(\boldsymbol{y})$ for $\boldsymbol{y} \in \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots\}$. Since both f and $f_{\boldsymbol{x}}$ are linear, it follows that $f_{\boldsymbol{x}}(\boldsymbol{y}) = f(\boldsymbol{y})$ for all $\boldsymbol{y} \in \operatorname{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots\} = c_{00}$. Then, since both f and $f_{\boldsymbol{x}}$ are continuous, it follows that $f_{\boldsymbol{x}}(\boldsymbol{y}) = f(\boldsymbol{y})$ for all $\boldsymbol{y} \in \overline{c_{00}} = c_0$: i.e. $f = f_{\boldsymbol{x}}$.

(iii) We have already shown that $\|f_{\boldsymbol{x}}\| \leqslant \|x\|_1$ and $\|x\|_1 \leqslant \|f\|$, so $\|f_{\boldsymbol{x}}\| = \|x\|_1$.

$\Diamond$

Theorem 10.2 *The dual of ℓ^1 is ℓ^∞. Specifically:*

(i) every $\boldsymbol{x} = (x_1, x_2, \ldots) \in \ell^\infty$ defines a continuous linear functional $f_{\boldsymbol{x}}$ on ℓ^1 by

$$f_{\boldsymbol{x}}(\boldsymbol{y}) = \sum_{i=1}^{\infty} x_i y_i \qquad (\boldsymbol{y} = (y_1, y_2, \ldots) \in \ell^1); \tag{13}$$

(ii) every continuous linear functional on ℓ^1 is of this form;

(iii) $\|f_{\boldsymbol{x}}\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \ell^\infty$, (where $\|f_{\boldsymbol{x}}\|$ is the norm of the linear functional and $\|\boldsymbol{x}\|$ is the ℓ^∞-norm of $\boldsymbol{x}$).

The proof, which is very similar to that of the previous theorem, is left as an exercise

Our next major example is Hilbert space: we shall show that the dual of a Hilbert space is itself. However, in order to do this, we need some more tools of Hilbert space theory. Specifically, we need the notion of *orthogonal projection* onto a given closed subspace. This projection is just the mapping taking a point to the nearest point in the subspace. But is there a nearest point? This is not immediately clear.

Lemma 10.3 *Let H be a Hilbert space and let $K \neq \{0\}$ be a closed linear subspace of H. Then there is a mapping $P \in \mathcal{B}(H)$ such that:*

(i) for every $x \in H$, Px is the unique nearest point in K to x;

(ii) $x - Px$ is orthogonal to every $y \in K$ (we write $(x - Px) \perp K$), and Px is the unique point in K with this property;

(iii) $P^2 = P$;

(iv) $\|P\| = 1$;

(v) $\langle x, Py\rangle = \langle Px, y\rangle$ for all $x, y \in H$ ('P is self-adjoint*').*

The mapping P is called the **orthogonal projection** of H onto K. (Actually, only part of this lemma is needed for our present purposes, but orthogonal projections are so important that it is worth explaining them fully.)

Proof. We first describe the construction of Px. The set of numbers $\{\|x - y\| : y \in K\}$ is bounded below (by 0), so it has a greatest lower bound

$$d = \inf\{\|x - y\| : y \in K\},$$

which we describe as the *distance of x from K*. We write $d = d(x, K)$.

Choose a sequence (y_n) in K with

$$\|x - y_n\| < d + \frac{1}{n}.$$

For $a, b \in K$,

$$\begin{aligned} \langle a+b, a+b\rangle + \langle a-b, a-b\rangle &= \langle a,a\rangle + \langle b,a\rangle + \langle a,b\rangle + \langle b,b\rangle + \langle a,a\rangle - \langle b,a\rangle - \langle a,b\rangle + \langle b,b\rangle \\ &= 2\langle a,a\rangle + 2\langle b,b\rangle. \end{aligned}$$

That is,

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2 \qquad (a, b \in H). \tag{14}$$

(This is just the median theorem. Remember that the subspace of H spanned by a and b looks just like $\mathbb{R}^2$.)

Then

$$\begin{aligned}
\|y_p - y_q\|^2 &= \|(x - y_q) - (x - y_p)\|^2 \\
&= 2\|(x - y_q)\|^2 + 2\|(x - y_p)\|^2 - \|(x - y_q) + (x - y_p)\|^2 \\
&= 2\|(x - y_q)\|^2 + 2\|(x - y_p)\|^2 - 4\left\|x - \frac{y_q + y_p}{2}\right\|^2 \\
&\leqslant 2\left(d + \frac{1}{q}\right)^2 + 2\left(d + \frac{1}{p}\right)^2 - 4d^2, \qquad (15)
\end{aligned}$$

since $(y_q + y_p)/2 \in K$.

Given $\varepsilon > 0$, choose N such that

$$\left(d + \frac{1}{N}\right)^2 - d^2 < \frac{\varepsilon}{4}.$$

Then, for $p, q \geqslant N$, we have

$$\|y_p - y_q\| \leqslant 2\left(d + \frac{1}{N}\right)^2 + 2\left(d + \frac{1}{N}\right)^2 - 4d^2 < 4\,\frac{\varepsilon}{4} = \varepsilon.$$

Thus (y_n) is a Cauchy sequence in K. But K is a closed subset of the complete space H, so K is complete, so (y_n) converges to some $y \in K$. Write $Px = y$.

Then, by the continuity of the norm, we have

$$d + \frac{1}{n} > \|x - y_n\| \to \|x - Px\| \geqslant d,$$

so $\|x - Px\| = d$; i.e. Px is **a** nearest point in K to x.

We now show that if u is a nearest point in K to x, then $(x - u) \perp K$. If $y \in K$ with $\|y\| = 1$, let $z = u + \langle x - u, y\rangle y$. Then

$$\begin{aligned}
d^2 &\leqslant \|x - z\|^2 \\
&= \|x - u - \langle x - u, y\rangle y\|^2 \\
&= \langle x - u - \langle x - u, y\rangle y, x - u - \langle x - u, y\rangle y\rangle \\
&= \langle x - u, x - u\rangle - \langle x - u, \langle x - u, y\rangle y\rangle - \langle\langle x - u, y\rangle y, x - u\rangle + \langle\langle x - u, y\rangle y, \langle x - u, y\rangle y\rangle \\
&= \|x - u\|^2 - \overline{\langle x - u, y\rangle}\langle x - u, y\rangle - \langle x - u, y\rangle\langle y, x - u\rangle + |\langle x - u, y\rangle|^2\,\|y\|^2 \\
&= d^2 - 2|\langle x - u, y\rangle|^2 + |\langle x - u, y\rangle|^2\,\|y\|^2 \\
&= d^2 - |\langle x - u, y\rangle|^2.
\end{aligned}$$

Therefore $\langle x - u, y\rangle = 0$. Hence, this is true for all $y \in K$; i.e. $(x - u) \perp K$. Thus $(x - Px) \perp K$.

If $u, v \in K$ such that $(x - u) \perp K$ and $(x - v) \perp K$, then, since $y = u - v \in K$, we have

$$\begin{aligned}
\|u - v\|^2 &= \langle u - v, u - v\rangle \\
&= \langle x - v, u - v\rangle - \langle x - u, u - v\rangle \\
&= 0,
\end{aligned}$$

so $u = v$. Thus Px is the unique point in K with $(x - Px) \perp K$, which proves (ii). Since every nearest point in K to x has this property, Px is the only nearest point in K to x, which proves (i).

We have still to prove that $P \in \mathcal{B}(H)$. Let $x_1, x_2 \in H$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then, for all $y \in K$,

$$\langle(\lambda_1 x_1 + \lambda_2 x_2) - (\lambda_1 Px_1 + \lambda_2 Px_2), y\rangle = \lambda_1\langle x_1 - Px_1, y\rangle + \lambda_2\langle x_2 - Px_2, y\rangle = 0,$$

so, by (ii),

$$P(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 P x_1 + \lambda_2 P x_2.$$

To show that P is continuous, we observe that, for all $x \in H$,

$$\begin{aligned} \|Px\|^2 &= \langle Px, Px\rangle \\ &= \langle x, Px\rangle - \langle x - Px, Px\rangle \\ &= \langle x, Px\rangle, \text{ by (ii), since } Px \in K. \end{aligned}$$

Therefore

$$\|Px\|^2 = |\langle x, Px\rangle| \leqslant \|x\|\,\|Px\|,$$

by the Cauchy-Schwarz inequality. Hence $\|Px\| \leqslant \|x\|$. Thus $P \in \mathcal{B}(H)$ with $\|P\| \leqslant 1$.

Now $P^2 x = P(Px)$ is the nearest point in K to Px, which must be Px itself, since $Px \in K$. Thus $P^2 x = Px$ for all $x \in H$, which proves (iii). However, this observation also gives $\|Py\| = \|y\|$ when $y \in P(H) = K$, so $\|P\| \geqslant 1$, which completes the proof of (iv). (This is where we need $K \neq \{0\}$.)

Finally, for all $x, y \in H$, we have $Px, Py \in K$, so

$$\begin{aligned} \langle x, Py\rangle &= \langle x - Px, Py\rangle + \langle Px, Py\rangle \\ &= \langle Px, Py\rangle, \text{ since } (x - Px) \perp K, \end{aligned}$$

Likewise $\langle y, Px\rangle = \langle Py, Px\rangle$, so

$$\langle Px, y\rangle = \overline{\langle y, Px\rangle} = \overline{\langle Py, Px\rangle} = \langle Px, Py\rangle = \langle x, Py\rangle,$$

which proves (v). $\Diamond$

Theorem 10.4 *The dual of a Hilbert space 'is' itself. Specifically: if H is a Hilbert space, then*

(i) every $x \in H$ defines a continuous linear functional f_x on H by

$$f_x(y) = \langle y, x\rangle \qquad (y \in H); \tag{16}$$

(ii) every continuous linear functional on H is of this form;

(iii) $\|f_x\| = \|x\|$ for all $x \in H$, (where $\|f_x\|$ is the norm of the linear functional and $\|x\|$ is the Hilbert space norm of x).

Proof.

(i) Clearly (16) defines a linear functional on H, and

$$|f_x(y)| = |\langle y, x\rangle| \leqslant \|x\|\,\|y\| \qquad (x, y \in H)$$

by the Cauchy-Schwarz inequality, so f_x is continuous with $\|f_x\| \leqslant \|x\|$.

(ii) Given a continuous linear functional f on H, if $f = 0$, then $f = f_x$ for $x = 0$. Otherwise, let $K = \ker f$. Then K is closed. Let P be the orthogonal projection of H onto K. Then choose $z \in H$ with $f(z) = 1$. For all $y \in H$, we have

$$f(f(y)z) = f(y)f(z) = f(y),$$

so $f(y - f(y)z) = 0$, i.e. $y - f(y)z \in K$, so

$$\langle y - f(y)z, z - Pz\rangle = 0$$

so

$$\langle y, z - Pz\rangle = \langle f(y)z, z - Pz\rangle = f(y)\langle z, z - Pz\rangle = f(y)\langle z - Pz, z - Pz\rangle = f(y)\|z - Pz\|^2.$$

Therefore $f(y) = \langle y, x\rangle = f_x(y)$ where $x = \|z - Pz\|^{-2}(z - Pz)$.

(iii) We have shown that $\|f_x\| \leqslant \|x\|$. For the reverse inequality, we observe that

$$|f_x(x)| = |\langle x, x\rangle| = \|x\|^2,$$

which shows that $\|f_x\| \geqslant \|x\|$.

$\Diamond$

Remark 10.5 Saying that H' 'is' H here is a bit misleading, in that the mapping $x \mapsto f_x : H \to H'$ is a conjugate-linear isometry: i.e.

$$f_{\lambda x+\mu y} = \overline{\lambda} f_x + \overline{\mu} f_y.$$

If we look at the specific Hilbert space ℓ^2 we can define

$$\phi_{\boldsymbol{x}}(\boldsymbol{y}) = \sum_{n=1}^{\infty} x_n y_n$$

for

$$\boldsymbol{x} = (x_1, x_2, x_3, \ldots), \quad \boldsymbol{y} = (y_1, y_2, y_3, \ldots)$$

in ℓ^2. Using the notation

$$\overline{\boldsymbol{x}} = (\overline{x_1}, \overline{x_2}, \overline{x_3}, \ldots)$$

then every $f \in H'$ is of the form

$$f = f_{\boldsymbol{x}} = \phi_{\overline{\boldsymbol{x}}}.$$

The mapping

$$\boldsymbol{x} \mapsto \phi_{\boldsymbol{x}} : \ell^2 \to (\ell^2)'$$

is then a linear isometry of ℓ^2 onto ℓ^2.

In fact, every Hilbert space H looks rather like ℓ^2 (to be precise, every separable Hilbert space is isomorphic to ℓ^2 as an inner product space), and so there is always a linear isometry between H and H'.

The previous theorem, with the above remark, tells us that the dual of ℓ^2 is ℓ^2 in a natural way. We have also seen (as an exercise) that $(\ell^1)' = \ell^\infty$ in the same natural way. These are special cases of the following general result about ℓ^p.

Theorem 10.6 *For $1 \leqslant p \leqslant \infty$, define the* conjugate index q *by*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

with the convention $1/\infty = 0$. Then

$$(\ell^p)' = \ell^q \qquad (1 \leqslant p < \infty),$$

in a natural way. As before, this means:

(i) every $\boldsymbol{x} = (x_1, x_2, \ldots) \in \ell^q$ defines a continuous linear functional $f_{\boldsymbol{x}}$ on ℓ^p by

$$f_{\boldsymbol{x}}(\boldsymbol{y}) = \sum_{i=1}^{\infty} x_i y_i \qquad (\boldsymbol{y} = (y_1, y_2, \ldots) \in \ell^p);$$

(ii) every continuous linear functional on ℓ^p is of this form;

(iii) $\|f_{\boldsymbol{x}}\| = \|\boldsymbol{x}\|_q$ for all $\boldsymbol{x} \in \ell^q$, (where $\|f_{\boldsymbol{x}}\|$ is the norm of the linear functional and $\|\boldsymbol{x}\|_q$ is the ℓ^q-norm of $\boldsymbol{x}$).

We omit the proof.

The cases we have already seen are $p = 1$, $q = \infty$, and $p = q = 2$. The other significant case is $p = \infty$, $q = 1$, but you will note that this has been excluded from the theorem.

It is certainly true that $\boldsymbol{x} \mapsto f_{\boldsymbol{x}}$ as above maps ℓ^1 linearly and isometrically into $(\ell^\infty)'$, but not every $f \in (\ell^\infty)'$ is of the form $f_{\boldsymbol{x}}$ for an $\boldsymbol{x} \in \ell^1$.

Theorem 10.7 *There exists $f \in (\ell^\infty)'$ which is not of the form $f_{\boldsymbol{x}}$ for an $\boldsymbol{x} \in \ell^1$.*

Proof. Let

$$c = \{\boldsymbol{x} \in \ell^\infty : \lim_{n\to\infty} x_n \text{exists}\},$$

define a continuous linear functional f on c by

$$f(\boldsymbol{x}) = \lim_{n\to\infty} x_n,$$

and extend it to a continuous linear functional f on the whole of ℓ^∞ by the Hahn–Banach Theorem.

Define $\boldsymbol{e}_n \in c$ $(n = 1, 2, 3, \ldots)$ by

$$(\boldsymbol{e}_n)_i = \begin{cases} 1 & (i = n) \\ 0 & (i \neq n). \end{cases}$$

If we had $f = f_{\boldsymbol{x}}$, then we should have $x_n = f(\boldsymbol{e}_n) = 0$, for all n, so $\boldsymbol{x} = \boldsymbol{0}$, so $f = 0$: contradiction. $\Diamond$

Using these sort of ideas, one can show that $(\ell^\infty)'$ is very big: one can show that it is non-separable. Since ℓ^1 is separable it follows that not only is $(\ell^\infty)'$ not isomorphic to ℓ^1 in this natural way: it is not isomorphic to ℓ^1 by any isomorphism.

The following definition is long overdue.

Definition 10.8 An **isomorphism** between two normed spaces E and F is a bijective linear map $T : E \to F$ such that both T and T^{-1} are continuous. If an isomorphism exists, we say that E and F are **isomorphic** normed spaces. If $\|.\|_1$ and $\|.\|_2$ are two norms on a space E, we say they are **equivalent** if the normed spaces $(E, \|.\|_1)$ and $(E, \|.\|_2)$ are isomorphic. This is equivalent to saying that there are constants $K, L > 0$ such that

$$K\|x\|_1 \leqslant \|x\|_2 \leqslant L\|x\|_1 \qquad (x \in E).$$

Remark 10.9 In the final chapter, we shall look at the case when E is a finite-dimensional normed space, and we shall show that E' is isomorphic to E.

11 Biduals and completions

Definition 11.1 The **bidual** E'' of a normed space E is simply the dual of the dual: $E'' = (E')'$.

Examples 11.2 In the following, '=' means that there is a natural isometric isomorphism between the spaces concerned.

(i) For $1 < p < \infty$, we have $(\ell^p)'' = \ell^p$ since $(\ell^p)' = \ell^q$ and $(\ell^q)' = \ell^p$, where $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) We have $(c_0)' = \ell^1$ and $(\ell^1)' = \ell^\infty$, so $(c_0)'' = \ell^\infty$.

The key fact about the bidual is the following.

Theorem 11.3 *For every normed space E, there is a natural isometric linear embedding ν of E into E'' given by*

$$(\nu(x))(f) = f(x) \qquad (x \in E,\ f \in E').$$

Proof. It is easy to check that $\nu(x)$ as defined above is a linear functional on E'. It is a continuous linear functional, of norm at most $\|x\|$ because

$$|(\nu(x))(f)| = |f(x)| \leqslant \|f\|\,\|x\| \qquad (f \in E').$$

To see that ν is an isometry, we recall Corollary 9.7:

$$\|x\| = \sup\{|f(x)| : f \in E',\ \|f\| \leqslant 1\}.$$

The right-hand side is

$$\sup\{|(\nu(x))(f)| : f \in E',\ \|f\| \leqslant 1\} = \|\nu(x)\|.$$

The proof that ν is linear is completely straightforward. $\Diamond$

If we start with a normed space E, not necessarily complete, then, by Corollary 7.8, the dual space E' is complete, as is E''. Thus, the previous theorem gives us an embedding of E into the complete space E''. Let

$$\tilde{E} = \overline{\nu(E)},$$

the closure in E'' of the image of E under this embedding.

Theorem 11.4 *For every normed space E there is a Banach space $\tilde{E}$ and an isometric embedding $\nu : E \to \tilde{E}$ such that $\nu(E)$ is dense in $\tilde{E}$. Further, $\tilde{E}$ is unique in the sense that if F is another Banach space such that there is an isometric embedding $\mu : E \to F$ such that $\mu(E)$ is dense in F, then there is an isometric isomorphism $\lambda : F \to E$ such that $\nu = \lambda\mu$.*

Proof. Let $\nu : E \to \tilde{E}$ be as above. Then ν is an isometric embedding and the definition of $\tilde{E}$ as the closure of $\nu(E)$ ensures that $\nu(E)$ is dense in $\tilde{E}$.

Now suppose that F and μ are as in the statement of the theorem. Then we can define an isometric isomorphism λ_1 of $\mu(E)$ onto $\nu(E)$ by $\lambda_1 = \nu\mu^{-1}$. Since $\mu(E)$ is dense in F, this extends by continuity to an isometric linear mapping $\lambda : F \to \tilde{E}$. Likewise, $\lambda_1^{-1} : \nu(E) \to \mu(E)$ extends to an isometric linear mapping $\phi : \tilde{E} \to F$. Now $\phi\lambda$ is a continuous map $F \to F$ which is the identity map on the dense subspace $\mu(E)$. Therefore, $\phi\lambda$ is the identity on F. Likewise, $\lambda\phi$ is the identity on $\tilde{E}$. Hence $\lambda : F \to \tilde{E}$ is an isomorphism. Since λ extends λ_1, we have $\nu = \nu\mu^{-1}\mu = \lambda_1\mu = \lambda\mu$. $\Diamond$

The space $\tilde{E}$, which is defined up to isometric isomorphism, is called the **completion** of E. It is usual to think of it as containing the space E, with ν being the inclusion map.

Definition 11.5 A normed space E is said to be **reflexive** if the map $\nu : E \to E''$ is an isomorphism; that is, if $\nu(E) = E''$.

Example 11.6 The space ℓ^p is reflexive for $1 < p < \infty$: it is easy to check that when $(\ell^p)'$ is identified with ℓ^q as above and $(\ell^q)'$ with ℓ^p, the mapping ν is just the identity mapping on ℓ^p.

However, for $p = 1$, Theorem 10.7 says that there is an $f \in (\ell^1)'' \setminus \nu(\ell^1)$, so ℓ^1 is not reflexive. In fact, $(\ell^1)'' = (\ell^\infty)'$ which is non-separable, so we cannot have ℓ^1 isomorphic to $(\ell^1)''$ either by ν or by any other isomorphism.

The space c_0 is not reflexive, since $(c_0)'' = \ell^\infty$ which is non-separable and so not isomorphic to c_0.

The space ℓ^∞ is also not reflexive.

[It is possible to have $E \cong E''$ without $\nu : E \cong E''$. It is also possible to have the weird situation where $\nu(E)$ is of codimension 1 in E''.]

12 The Uniform Boundedness Theorem

In Functional Analysis, there are three important theorems derived from Baire's Category Theorem: the Uniform Boundedness Theorem, The Open Mapping Theorem and the Closed Graph Theorem. First, we recall Baire's theorem, (from PMA344).

Theorem 12.1 (Baire) *Let (X, d) be a complete metric space. Let (F_n) be a sequence of closed subsets of X for which $\bigcup_{n=1}^{\infty} F_n = X$. Then at least one of the sets F_n has a non-empty interior: i.e. contains a non-empty open set.*

Definition 12.2 Let E, F be normed spaces, and $L \subseteq \mathcal{B}(E, F)$. The set L is *uniformly bounded* if there exists $M > 0$ such that $\|T\| \leqslant M$ for all $T \in L$. The set L is *pointwise bounded* if, for all $x \in E$ there exists $M_x > 0$ such that $\|Tx\| \leqslant M_x$ for all $T \in L$.

Theorem 12.3 (Uniform Boundedness Theorem) *If E, F are normed spaces, and $L \subseteq \mathcal{B}(E, F)$, and E is complete, then L is uniformly bounded if and only if it is pointwise bounded.*

Proof. The direction 'uniformly bounded implies pointwise bounded' is trivial: if $\|T\| \leqslant M$ for all $T \in L$, then $\|Tx\| \leqslant M\|x\|$ and $M_x = M\|x\|$ is the desired pointwise bound.

For the converse, let

$$B = \{x \in E : \|Tx\| \leqslant 1 \quad (T \in L)\}.$$

The fact that L is pointwise bounded implies that for all $x \in E$ there is a positive integer n such that

$$\|Tx\| \leqslant n \qquad (T \in L).$$

Therefore, for all $x \in E$, there exists n such that $n^{-1}x \in B$. That is,

$$E = \bigcup_{n=1}^{\infty} nB.$$

Now

$$B = \bigcap_{T \in L} T^{-1}(F_1).$$

Since the maps T are continuous, the sets $T^{-1}(F_1)$ are closed, so the set B is closed, so (since the maps $x \mapsto nx$ are homeomorphisms), the sets nB are all closed.

Thus we have the complete metric space E expressed as a countable union of closed sets. Baire's Category Theorem implies that at least one of these sets must have non-empty interior: say $\operatorname{int}(n_0 B) \neq \emptyset$. Since $x \mapsto n_0 x$ is a homeomorphism, $\operatorname{int}(B) \neq \emptyset$.

Therefore there exists $x_0 \in B$ and $\delta > 0$ such that

$$B(x_0; \delta) \subseteq B;$$

i.e., for all $x \in E_\delta^\circ$,

$$x_0 + x \in B \qquad \text{and} \qquad x_0 \in B;$$

i.e., for all $x \in E_\delta^\circ$ and all $T \in L$,

$$\|T(x_0 + x)\| \leqslant 1 \qquad \text{and} \qquad \|T(x_0)\| \leqslant 1;$$

so

$$\|T(x)\| \leqslant \|T(x_0 + x)\| + \|-T(x_0)\| \leqslant 1 + 1 = 2.$$

Therefore

$$\|Tx\| \leqslant \frac{2}{\delta} \qquad (\|x\| < 1,\ T \in L).$$

So $\|T\| \leqslant 2/\delta$ $(T \in L)$. So L is uniformly bounded. $\Diamond$

Corollary 12.4 *Let E be a Banach space. Then a set $L \subseteq E'$ is bounded if and only if $\{f(x) : f \in L\}$ is bounded in $\mathbb{C}$ for each $x \in E$.*

Proposition 12.5 *Let E be a Banach space, F a normed space, and (T_n) a sequence in $\mathcal{B}(E, F)$ such that $\lim_{n\to\infty} T_n(x)$ exists for each $x \in E$. Define $T : E \to F$ by*

$$T(x) = \lim_{n\to\infty} T_n(x) \qquad (x \in E).$$

Then the sequence (T_n) is bounded in $\mathcal{B}(E, F)$ and $T \in \mathcal{B}(E, F)$.

Proof. Clearly T is linear, since

$$\begin{aligned} T(\lambda x + \mu y) &= \lim_{n\to\infty} T_n(\lambda x + \mu y) \\ &= \lim_{n\to\infty} (\lambda T_n(x) + \mu T_n(y)) \\ &= \lambda \lim_{n\to\infty} T_n(x) + \mu \lim_{n\to\infty} T_n(y) \\ &= \lambda T(x) + \mu T(y). \end{aligned}$$

Let $L = \{T_n : n = 1, 2, 3, , \ldots\}$ and apply Theorem 12.3: for each $x \in E$, the set $\{T_n(x) : n = 1, 2, 3, \ldots\}$ is bounded (because convergent sequences in a normed space are bounded), so we can deduce from Theorem 12.3 that there is a number K such that $\|T_n\| \leqslant K$ for all n, which is our first conclusion. Then

$$\|T_n(x)\| \leqslant K\|x\| \qquad (x \in E,\ n = 1, 2, 3, \ldots).$$

Hence

$$\|T(x)\| = \left\|\lim_{n\to\infty} T_n(x)\right\| = \lim_{n\to\infty} \|T_n(x)\| \leqslant K\|x\| \qquad (x \in E),$$

So T is bounded: i.e. $T \in \mathcal{B}(E, F)$. $\Diamond$

Corollary 12.6 *Let E be a Banach space and let (f_n) be a sequence in E' such that $\lim_{n\to\infty} f_n(x)$ exists for each $x \in E$. Define $f : E \to \mathbb{C}$ by*

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (x \in E).$$

Then the sequence (f_n) is bounded in E' and $f \in E'$.

Exercise 12.7 Show that we can not add to the conclusions of this proposition that $f_n \to f$ in E'. (Hint: let E be a suitable space of sequences — you could choose c_0 or ℓ^p for some $p \in [1, \infty)$ — and let f_n be the linear functional associated with an element $(0, 0, \ldots, 0, 1, 0, \ldots)$ in E'.)

13 The Open Mapping and Closed Graph Theorems

The theorems in this section are the key to a deeper understanding of functional analysis. The Closed Graph Theorem in particular shows why you hardly every come across unbounded linear mapping defined on Banach spaces, though it's quite easy to find them defined on incomplete normed spaces. Most 'automatic continuity' results, in which mappings satisfying algebraic conditions are shown to be automatically continuous, rely on this theorem.

Definition 13.1 A mapping $T : X \to Y$ between two metric spaces is said to be **open** if, for every open set $G \subseteq X$ the image $T(G)$ is open in Y.

Remark 13.2 Let $T : E \to F$ be a linear mapping between normed spaces. Then T is open if and only if $T(E_1^\circ)$ is a neighbourhood of 0 in F. If T is open, then T is surjective.

Theorem 13.3 (Open Mapping Theorem) *If E, F are Banach spaces and $T \in \mathcal{B}(E, F)$ is surjective, then T is open.*

Proof. The proof is in two main stages.

1. We show first that $\overline{T(E_1^\circ)}$ is a neighbourhood of 0 in F: i.e. that $\delta F_1^\circ \subseteq \overline{T(E_1^\circ)}$ for some $\delta > 0$.

   We have

   $$F = \bigcup_{n=1}^{\infty} nT(E_1^\circ) \subseteq \bigcup_{n=1}^{\infty} n\overline{T(E_1^\circ)}.$$

   Since F is complete, we deduce from Baire's Category Theorem that some $n_0\overline{T(E_1^\circ)}$ has non-empty interior, say

   $$B(y_0; \varepsilon) \subseteq n_0\overline{T(E_1^\circ)}.$$

Thus, for every $y \in F_1^\circ$,

$$y_0 + \varepsilon y \in B(y_0; \varepsilon) \subseteq n_0 \overline{T(E_1^\circ)},$$

and

$$y_0 \in B(y_0; \varepsilon) \subseteq n_0 \overline{T(E_1^\circ)}.$$

Now E_1° is convex, so $T(E_1^\circ)$ is convex, so $\overline{T(E_1^\circ)}$ is convex, so $2n_0\overline{T(E_1^\circ)}$ is convex. Both $2(y_0+\varepsilon y)$ and $-2y_0$ are in $2n_0\overline{T(E_1^\circ)}$, so, by convexity it follows that

$$\varepsilon y = \frac{1}{2}\left(2(y_0 + \varepsilon y) + (-2y_0)\right) \in 2n_0\overline{T(E_1^\circ)}.$$

Thus we have shown that

$$\varepsilon F_1^\circ \subseteq 2n_0\overline{T(E_1^\circ)}.$$

So $\delta F_1^\circ \subseteq \overline{T(E_1^\circ)}$ where $\delta = \dfrac{\varepsilon}{2n_0}$.

2. We now show that $\overline{T(E_1^\circ)} \subseteq 2T(E_1^\circ)$.

   Let $y \in \overline{T(E_1^\circ)}$. Then there exists $x_1 \in E_1^\circ$ with

$$\|y - Tx_1\| < \frac{1}{2}\delta.$$

   By part 1,

$$y - Tx_1 \in \frac{1}{2}\delta F_1^\circ \subseteq \frac{1}{2}\overline{T(E_1^\circ)}.$$

   Applying this to the point $2(y - Tx_1) \in \overline{T(E_1^\circ)}$, we obtain $x_2 \in E_1^\circ$ with

$$\|2(y - Tx_1) - Tx_2\| < \frac{1}{2}\delta,$$

   i.e.

$$\|y - Tx_1 - \frac{1}{2}Tx_2\| < \frac{1}{4}\delta.$$

   Proceeding thus, we obtain a sequence of points $x_1, x_2, \ldots \in E_1^\circ$ such that

$$\left\| y - T\left(x_1 + \frac{1}{2}x_2 + \ldots + \frac{1}{2^{n-1}}x_n\right)\right\| < 2^{-n}\delta.$$

   Since the x_i are all in E_1° and E is Banach, the sum

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}x_n$$

   converges in E, to some point x, say, and

$$\|x\| = \lim_{n\to\infty}\left\|x_1 + \frac{1}{2}x_2 + \ldots + \frac{1}{2^{n-1}}x_n\right\| < 2.$$

   Now

$$Tx = \lim_{n\to\infty} T\left(x_1 + \frac{1}{2}x_2 + \ldots + \frac{1}{2^{n-1}}x_n\right),$$

   since T is continuous, so

$$\begin{aligned}\|y - Tx\| &= \lim_{n\to\infty}\left\|y - T\left(x_1 + \frac{1}{2}x_2 + \ldots + \frac{1}{2^{n-1}}x_n\right)\right\| \\ &= 0.\end{aligned}$$

   Thus $y = Tx$, and so $y \in 2T(E_1^\circ)$.

Putting parts 1 and 2 together, we see that

$$T\left(E_1^\circ\right) \supseteq \frac{\delta}{2} F_1^\circ,$$

so T is open. $\Diamond$

Theorem 13.4 (Banach's Isomorphism Theorem) *If E, F are Banach spaces and $T \in \mathcal{B}(E, F)$ is bijective, then T is an isomorphism.*

Proof. The Open Mapping Theorem shows that T is open, which, since T is bijective, is the same as saying that T^{-1} is continuous; so T is an isomorphism. $\Diamond$

Theorem 13.5 (Equivalence of comparable complete norms) *Suppose $\|.\|_1$ and $\|.\|_1$ are two complete norms on a space E which are* comparable*: that is, either (i) there exists $K > 0$ such that $\|x\|_1 \leqslant K\|x\|_2$ for all $x \in E$, or (ii) there exists $K > 0$ such that $\|x\|_2 \leqslant K\|x\|_1$ for all $x \in E$. Then $\|.\|_1$ and $\|.\|_1$ are equivalent.*

Proof. This is just Banach's Isomorphism Theorem applied to the identity mapping $(E, \|.\|_2) \to (E, \|.\|_1)$, in case (i), or to the identity $(E, \|.\|_1) \to (E, \|.\|_2)$, in case (ii). $\Diamond$

Examples 13.6

1. On the space $\mathbb{C}^n$ we have $\|.\|_\infty \leqslant \|.\|_1$ and these norms are equivalent.
2. On the space $C[0,1]$ we have $\|.\|_\infty \leqslant \|.\|_1$ and these norms are not equivalent, but whilst $\|.\|_\infty$ is a complete on $C[0,1]$, the norm $\|.\|_1$ is not.

The Closed Graph Theorem

It is very difficult to construct discontinuous linear mappings on Banach spaces.

Example 13.7 Let $E = C[0,1]$ with the usual sup norm. The linear functional

$$f(x) = x'\left(\frac{1}{2}\right)$$

is discontinuous since one can easily produce functions x of norm 1 whose derivatives at $1/2$ are arbitrarily large. Unfortunately, f is only defined on the subspace F of E consisting of those functions which are differentiable at the point $1/2$. This subspace is not closed (in fact it is dense in E), so it is not itself a Banach space. If we want to extend f to the whole of E, we can do so using a version of the Hahn-Banach Theorem without clause about the functionals being continuous. This modified Hahn-Banach Theorem — a purely algebraic result saying that linear functionals can be extended from subspaces — is easily proved by cutting all mention of norms and sublinear functionals out of the original proof; in fact, it is a much easier theorem. But — it still depends on the Axiom of Choice.

The above example is typical. Why do we always find ourselves using the Axiom of Choice? We can we not just write down a formula for an unbounded linear mapping? The Closed Graph Theorem tells us why.

Definition 13.8 If $T : X \to Y$ is any mapping between two sets, the **graph** of T is the set

$$\mathrm{Gr}T = \{(x, Tx) : x \in X\} \subseteq X \times Y.$$

Definition 13.9 If $T : X \to Y$ is a mapping between two metric spaces, we say T **has a closed graph** if $\mathrm{Gr}T$ is a closed subset of $X \times Y$; i.e.

$$(x_n, Tx_n) \to (x, y) \quad \Rightarrow \quad (x, y) \in \mathrm{Gr}T;$$

i.e.

$$x_n \to x \text{ and } Tx_n \to y \quad \Rightarrow \quad y = Tx.$$

Exercise 13.10 Show that if $T : X \to Y$ is bijective, then T has a closed graph if and only if T^{-1} has a closed graph.

Generally, a mapping you construct by writing down a simple formula will have a closed graph.

Theorem 13.11 *If $T : X \to Y$ is a continuous mapping between metric spaces, then T has a closed graph.*

Proof. If $x_n \to x$ and $Tx_n \to y$, then the continuity of T implies that $Tx_n \to Tx$ and uniqueness of limits gives us $Tx = y$. $\Diamond$

Theorem 13.12 (Closed Graph Theorem) *Let $T : E \to F$ be a linear mapping between two Banach spaces. If T has a closed graph, then T is continuous.*

Proof. If T has a closed graph, then $\mathrm{Gr}T$ is a closed linear subspace [exercise: check this] of the Banach space [Question Sheet 3, Question 2(iv)] $E \times F$. Therefore $\mathrm{Gr}T$ is a Banach space.

Consider the mapping $P : \mathrm{Gr}T \to E : (x, Tx) \mapsto x$. Then P is continuous ($\|P(x, Tx)\| = \|x\| \leqslant \max\{\|x\|, \|Tx\|\} = \|(x, Tx)\|$) and bijective. Likewise, the mapping $P : \mathrm{Gr}T \to E : (x, Tx) \mapsto Tx$ is continuous.

Banach's Isomorphism Theorem applied to P tells us that $P^{-1} : x \mapsto (x, Tx)$ is continuous. Therefore $T = QP^{-1}$ is continuous. $\Diamond$

Corollary 13.13 (Equivalence of complete norms with a common weaker topology) *Suppose $\|.\|_1$ and $\|.\|_1$ are two complete norms on a space E and suppose that there is a metric d on E such that*

$$\|x_n - x\|_i \to 0 \quad \Rightarrow \quad d(x_n, x) \to 0 \qquad (i = 1, 2).$$

Then $\|.\|_1$ and $\|.\|_1$ are equivalent.

This is a strengthening of Theorem 13.5, to which it reduces in the case when $d(x, y) \equiv \|x - y\|_i$ for $i = 1$ or 2.

Proof. Given the hypotheses, the identity mapping $I : (E, \|.\|_1) \to (E, \|.\|_2)$ has a closed graph, because if $x_n \to x$ in $\|.\|_1$ and $Ix_n \to y$, i.e. $x_n \to y$ in $\|.\|_2$, then $\|x_n - x\|_1 \to 0$ and $\|x_n - y\|_2 \to 0$, so $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$, so $x = y$ by the uniqueness of limits in the metric space (E, d); i.e. $Ix = y$.

The Closed Graph Theorem then implies that I is continuous. By symmetry, I^{-1} is continuous, so the norms are equivalent. $\Diamond$

14 Finite-dimensional normed spaces

In this chapter, we show that the theory of finite-dimensional normed spaces is fairly trivial. In particular, we shall show that all norms on a finite-dimensional space are equivalent. We begin with some preliminaries concerning equivalent norms. We recall

Definition 14.1 An **isomorphism** between two normed spaces E and F is a bijective linear map $T : E \to F$ such that both T and T^{-1} are continuous. If an isomorphism exists, we say that E and F are **isomorphic** normed spaces. If $\|.\|_1$ and $\|.\|_2$ are two norms on a space E, we say they are **equivalent** if the normed spaces $(E, \|.\|_1)$ and $(E, \|.\|_2)$ are isomorphic. This is equivalent to saying that there are constants $K, L > 0$ such that

$$K\|x\|_1 \leqslant \|x\|_2 \leqslant L\|x\|_1 \qquad (x \in E).$$

If $I : (E, \|.\|_1) \to (E, \|.\|_2)$ is the identity operator, then the constants appearing in the equivalent formulation are $K = \|I^{-1}\|^{-1}$ and $L = \|I\|$: the proof is left to the reader. Since continuity of I and of I^{-1} can be characterized by convergent sequences mapping to convergent sequences or by inverse images of open/closed sets being open/closed, respectively, we have the following easy proposition.

Proposition 14.2 *If $\|.\|_1$ and $\|.\|_2$ are two norms on a space E, the following are equivalent:*

(a) $\|.\|_1$ and $\|.\|_2$ are equivalent, i.e. the identity map $(E, \|.\|_1) \to (E, \|.\|_2)$ is an isomorphism;

(b) there are constants $K, L > 0$ such that

$$K\|x\|_1 \leqslant \|x\|_2 \leqslant L\|x\|_1 \qquad (x \in E);$$

(c) a sequence (x_n) in E converges in the norm $\|.\|_1$ to a point $x \in E$ if and only if $x_n \to x$ in the norm $\|.\|_2$;

(d) a set $G \subseteq E$ is open in $(E, \|.\|_1)$ if and only if it is open in $(E, \|.\|_2)$;

(e) a set $F \subseteq E$ is closed in $(E, \|.\|_1)$ if and only if it is closed in $(E, \|.\|_2)$.

We now that completeness is not preserved by homeomorphisms between metric spaces, but it is preserved by linear isomorphisms. Essentially, the linear structure enables us to compare smallness at different points by means of translations.

Proposition 14.3 *If $T : E \to F$ is an isomorphism between normed spaces and E is complete, then so is F.*

Proof. Let y_n be a Cauchy sequence in F. Since T is surjective, there is a sequence (x_n) in E with $y_n = Tx_n$ for all n. Now given $\varepsilon > 0$, there exists N such that

$$\|y_p - y_q\| < \frac{\varepsilon}{\|T^{-1}\|} \qquad (p, q \geqslant N).$$

Then

$$\begin{aligned} \|x_p - x_q\| &\leqslant \|T^{-1}\| \, \|T(x_p - x_q)\| \\ &\leqslant \|T^{-1}\| \, \|Tx_p - Tx_q\| \\ &= \|T^{-1}\| \, \|y_p - y_q\| \\ &< \varepsilon. \end{aligned}$$

Thus (x_n) is a Cauchy sequence in E. But E is complete, so $x_n \to x$ for some $x \in E$. Since T is continuous, it follows that $y_n = Tx_n \to Tx \in F$. This proves that F is complete. $\Diamond$

Corollary 14.4 *If $\|.\|_1$ and $\|.\|_2$ are two norms on a space E and $(E, \|.\|_1)$ is complete, then so is $(E, \|.\|_2)$.*

Theorem 14.5 *Let E be a normed space of finite dimension n. Then*

(a) all linear functionals on E are continuous;

(b) E is isomorphic to the space $(\mathbb{C}^n, \|.\|_\infty)$;

(c) E is complete.

Proof. We prove this by induction on n.

For $n = 0$ we have $E = \{0\}$ and the results are trivial.

Suppose the theorem holds for dimension $n - 1$ and $\dim E = n$. If f is a non-zero linear functional on E, then $\ker f$ is a linear subspace of E of dimension $n - 1$. Applying the induction hypothesis to $\ker f$, we obtain that $\ker f$ is complete. Therefore $\ker f$ is closed in E. Therefore, by Theorem 7.9, f is continuous, which proves (a) for dimension n.

Now (a) implies (b), for if we choose a basis $\{x_1, \ldots, x_n\}$ of E, then for all $x \in E$ we can write

$$x = f_1(x)x_1 + \ldots + f_n(x)x_n,$$

where the f_i are uniquely defined, linear functionals. By (a), the f_i are continuous. Define

$$T : x \mapsto (f_1(x), \ldots, f_n(x)) : (E, \|.\|) \to (\mathbb{C}^n, \|.\|_\infty).$$

Then T is continuous, since all the f_i are continuous. But

$$T^{-1} : (\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 x_1 + \ldots + \lambda_n x_n$$

is continuous, by the continuity of algebraic operations in normed spaces. Therefore T is an isomorphism.

Further, (b) implies (c), since $(\mathbb{C}^n, \|.\|_\infty)$ is complete and isomorphisms preserve completeness (Proposition 14.3).

Thus we have deduced from the induction hypothesis that (a), (b), and (c) hold for dimension n. The desired result follows by induction. $\Diamond$

Corollary 14.6 *All norms on a finite-dimensional space are equivalent.*

Proof. If $\|.\|_1$ and $\|.\|_2$ are two norms on a space E of dimension n, we have shown that both $(E, \|.\|_1)$ and $(E, \|.\|_2)$ are isomorphic to $(\mathbb{C}^n, \|.\|_\infty)$, and so they are isomorphic to each other: i.e. the two norms are equivalent. $\Diamond$

Corollary 14.7 *Every finite-dimensional subspace of a normed space is closed.*

Proof. Let F be a finite-dimensional subspace of a normed space E. Then F is complete. Every complete subset of a metric space is closed. Therefore F is closed in E. $\Diamond$

Theorem 14.8 *Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be normed spaces, with E finite-dimensional. Then every linear mapping $T : E \to F$ is continuous.*

Proof. We define a new norm $\|.\|'$ on E by

$$\|x\|' = \|x\|_E + \|Tx\|_F \qquad (x \in E).$$

We check that it is a norm:

(a)

$$\begin{aligned} \|x+y\|' &= \|x+y\|_E + \|T(x+y)\|_F \\ &= \|x+y\|_E + \|Tx+Ty\|_F \\ &\leqslant \|x\|_E + \|y\|_E + \|Tx\|_F + \|Ty\|_F \\ &= \|x\|' + \|y\|'. \qquad (x, y \in E) \end{aligned}$$

(b)

$$\begin{aligned} \|\lambda x\|' &= \|\lambda x\|_E + \|T(\lambda x)\|_F \\ &= \|\lambda x\|_E + \|\lambda Tx\|_F \\ &\leqslant |\lambda| \, \|x\|_E + |\lambda| \, \|Tx\|_F \\ &= |\lambda| \, \|x\|'. \qquad (x \in E, \ \lambda \in \mathbb{C}) \end{aligned}$$

(c) $\|x\|' \geqslant \|x_E\| > 0$ when $x \neq 0$.

Thus $\|.\|'$ is another norm on E, so, by 14.6, it is equivalent to $\|.\|_E$. Therefore there is a constant K such that

$$\|x\|' \leqslant K\|x\|_E \qquad (x \in E),$$

so

$$\|Tx\|_F \leqslant \|x\|' \leqslant K\|x\|_E \qquad (x \in E),$$

so T is continuous. $\Diamond$

Note that it is **not** true that every mapping **from** an infinite-dimensional normed space, or even a Banach space, **into** a finite-dimensional space is continuous.

Example 14.9 Let $E = c_{00}$ and define a linear functional f on E by

$$f(\boldsymbol{x}) = \sum_{i=1}^{\infty} x_i \qquad (\boldsymbol{x} = (x_1, x_2, \ldots \in E).$$

Then f is certainly unbounded; if $\boldsymbol{x}^{(n)} = (1, 1, \ldots 1, 0, 0, \ldots)$ with n 1s, then $\|\boldsymbol{x}^{(n)}\|_\infty = 1$ but $f(x) = n$.

We can extend f to an unbounded linear functional on the Banach space c_0 by using a version of the Hahn-Banach Theorem without the control on the norm of the linear functional — just delete all references to sublinear functionals and norms throughout the statement and proof of the Hahn-Banach Theorem.