PMA445 Functional Analysis 2004
Question Sheet 1

To be handed in on Tuesday 24 February.

1. Let $a, b > 0$. Let $E = \mathbb{C}^2$. Show that the function

$$\|x\| = a|x_1| + b|x_2| \qquad (x = (x_1, x_2) \in \mathbb{C})$$

is a norm on E. (The generalization of this to spaces of infinite sequences $x = (x_1, x_2, x_3, \ldots)$ produces 'weighted sequence spaces'.)

2. Let $(E, \|.\|)$ be a normed space; we attempt to define a new norm $\|.\|'$ on E by

$$\|x\|' := \frac{\|x\|}{1 + \|x\|} \qquad (x \in E).$$

Show that $\|.\|'$ always satisfies two of the three axioms for a norm, but that it can fail to satisfy the remaining axiom, even in the case $(E, \|.\|) = (\mathbb{C}, |.|)$.

3. A subset C of a vector space E (over $\mathbb{C}$) is said to be *convex* if whenever $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$ in $\mathbb{R}$ then $\lambda x + (1 - \lambda)y \in C$. Let E be a normed space.

(i) Let $K > 0$. Prove that the set $E_K = \{x \in E : \|x\| \leqslant K\}$ is convex.

(ii) Let C be a convex subset of E. Prove that $\overline{C}$ is also convex.

PMA445 Functional Analysis 2004
Question Sheet 1: Solutions

1. *Let $a, b > 0$. Let $E = \mathbb{C}^2$. Show that the function*

$$\|x\| = a|x_1| + b|x_2| \qquad (x = (x_1, x_2) \in \mathbb{C})$$

is a norm on E.

We check the axioms:

(i) For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in E, we have $x + y = (x_1 + y_1, x_2 + y_2)$, so

$$\|x + y\| = a|x_1 + y_1| + b|x_2 + y_2| \leqslant a|x_1| + a|y_1| + b|x_2| + b|y_2| = \|x\| + \|y\|.$$

(ii) For $x = (x_1, x_2) \in E$ and $\lambda \in \mathbb{C}$, we have $\lambda x = (\lambda x_1, \lambda x_2)$, so

$$\|\lambda x\| = a|\lambda x_1| + b|\lambda x_2| = a|\lambda||x_1| + b|\lambda||x_2| = |\lambda|\,\|x\|.$$

(iii) For $x = (x_1, x_2) \in E$, we have

$$\begin{aligned} \|x\| = 0 \quad &\iff \quad a|x_1| + b|x_2| = 0 \\ &\iff \quad |x_1| = |x_2| = 0, \text{ since } a, b > 0, \\ &\iff \quad x = 0. \end{aligned}$$

2. *Let $(E, \|.\|)$ be a normed space; we attempt to define a new norm $\|.\|'$ on E by*

$$\|x\|' := \frac{\|x\|}{1 + \|x\|} \qquad (x \in E).$$

Show that $\|.\|'$ always satisfies two of the three axioms for a norm, but that it can fail to satisfy the remaining axiom, even in the case $(E, \|.\|) = (\mathbb{C}, |.|)$.

We check the axioms:

(i) For $x, y \in E$, we have

$$\begin{aligned} \|x + y\|' &= \frac{\|x + y\|}{1 + \|x + y\|} \\ &\leqslant \frac{\|x\|}{1 + \|x\|} + \frac{\|y\|}{1 + \|y\|} \qquad (*) \\ &= \|x\|' + \|y\|', \end{aligned}$$

where (*) will hold iff

$$\|x + y\|(1 + \|x\|)(1 + \|y\|) \leqslant \|x\|(1 + \|y\|)(1 + \|x + y\|) + \|y\|(1 + \|x\|)(1 + \|x + y\|),$$

i.e. iff

$$\|x + y\|(1 + \|x\| + \|y\| + \|x\|\,\|y\|) \leqslant (\|x\| + \|y\|)(1 + \|x + y\|) + 2\|x\|\,\|y\|(1 + \|x + y\|),$$

i.e. iff

$$\|x + y\|(1 + \|x\| + \|y\|) \leqslant (\|x\| + \|y\|)(1 + \|x + y\|) + \|x\|\,\|y\|(2 + \|x + y\|),$$

i.e. iff

$$\|x+y\| \leqslant \|x\|+\|y\|+\|x\|\,\|y\|(2+\|x+y\|),$$

which holds since

$$\|x+y\| \leqslant \|x\|+\|y\| \quad \text{and} \quad \|x\|\,\|y\|(2+\|x+y\|) \geqslant 0.$$

(ii) Let $(E, \|.\|) = (\mathbb{C}, |.|)$ and let $x = 1 \in E$, $\lambda = 2 \in \mathbb{C}$. Then

$$\|x\| = 1, \quad \|x\|' = \frac{1}{2}, \quad \|\lambda x\| = 2, \quad \|\lambda x\|' = \frac{2}{3} \neq 1 = |\lambda|\,\|x\|'.$$

Thus the second axiom fails. N.B. We show failure of the this axiom by producing a **very specific counterexample.**

(iii) For $x \in E$, we have

$$\begin{aligned} \|x\|' = 0 \quad &\iff \quad \frac{\|x\|}{1+\|x\|} = 0 \\ &\iff \quad \|x\| = 0 \\ &\iff \quad x = 0, \end{aligned}$$

since $\|.\|$ is a norm.

3. *A subset C of a vector space E (over $\mathbb{C}$) is said to be convex if whenever $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$ in $\mathbb{R}$ then $\lambda x + (1-\lambda)y \in C$. Let E be a normed space.*

(i) Let $K > 0$. Prove that the set $E_K = \{x \in E : \|x\| \leqslant K\}$ is convex.

(ii) Let C be a convex subset of E. Prove that $\overline{C}$ is also convex.

(i) Let $0 \leqslant \lambda \leqslant 1$ in $\mathbb{R}$, and let $x, y \in E_K$. Then

$$\begin{aligned} \|\lambda x + (1-\lambda)y\| \quad &\leqslant \quad \lambda\|x\| + (1-\lambda)\|y\| \quad (\text{ noting that } 1-\lambda \geqslant 0) \\ &\leqslant \quad \lambda K + (1-\lambda)K = K, \end{aligned}$$

so that $\lambda x + (1-\lambda)y \in E_K$.

(ii) Let $0 \leqslant \lambda \leqslant 1$ in $\mathbb{R}$, and let $x, y \in \overline{C}$. So choose sequences (x_n) and (y_n) in C converging to x and y respectively. Using the continuity of algebraic operations,

$$\lambda x_n + (1-\lambda)y_n \to \lambda x + (1-\lambda)y.$$

Since $\lambda x_n + (1-\lambda)y_n \in C$, we have $\lambda x + (1-\lambda)y \in \overline{C}$, as required.

PMA445 Functional Analysis 2004
Question Sheet 2

Not to be handed in. Solutions will be distributed on Tuesday 2 March.

1. Let E be a normed space, and $E_1 = \{x \in E : \|x\| \leqslant 1\}$. Prove that E is complete if and only if E_1 is complete. [Hint: use the fact that every Cauchy sequence is bounded.] Deduce that if E_1 is compact then E is complete.

2. Let E be a normed space. Let $K > 0$. Prove that E_K is closed, and E_K° is open, in E.

3. Let E be a normed space. Let $K > 0$. Prove that $\overline{E_K^\circ} = E_K$.

PMA445 Functional Analysis 2004
Question Sheet 2: Solutions

1. *Let E be a normed space, and $E_1 = \{x \in E : \|x\| \leqslant 1\}$. Prove that E is complete if and only if E_1 is complete. [Hint: use the fact that every Cauchy sequence is bounded.] Deduce that if E_1 is compact then E is complete.*

   Since E_1 is a closed subset of the (metric) space E, if E is complete then so is E_1.

   So now let E_1 be complete. Let (x_n) be a Cauchy sequence in E. Then (x_n) is bounded; i.e. there exists $M > 0$ so that $\|x_n\| \leqslant M$ for all $n \in \mathbb{N}$. Let $y_n = \frac{1}{M}x_n$, for each n. Then $y_n \in E_1$. Given $\varepsilon > 0$ we can find N so that $\|x_n - x_m\| < M\varepsilon$ whenever $n, m \geqslant N$. So then $\|y_n - y_m\| = \frac{1}{M}\|x_n - x_m\| < \varepsilon$ whenever $n, m \geqslant N$. So (y_n) is Cauchy and so convergent, to y, say. Let $x = My$. Then, by the continuity of scalar multiplication, (x_n) converges to x. Thus E is complete, since every Cauchy sequence converges.

   Finally, since compact implies complete (in any metric space), if E_1 is compact, then E_1 is complete, so E is complete.

2. *Let E be a normed space. Let $K > 0$. Prove that E_K is closed, and E_K° is open, in E.*

   Let (x_n) be a sequence in E_K converging to a point $x \in E$. Then $\|x_n\| \leqslant K$ for each $n \in \mathbb{N}$. By the continuity of $x \mapsto \|x\|$, we can conclude that $\|x_n\| \to \|x\|$, so that it follows that $\|x\| \leqslant K$ also, i.e. $x \in E_K$. Thus E_K is closed.

   Now let $x \in E_K^\circ$. Then $\|x\| < K$. Let $\varepsilon = K - \|x\|$, so $\varepsilon > 0$.

   If $y \in E$ with $\|y - x\| < \varepsilon$, then

   $$\|y\| = \|(y - x) + x\| \leqslant \|y - x\| + \|x\| < \varepsilon + \|x\| = K - \|x\| + \|x\| = K,$$

   so that $y \in E_K^\circ$, by definition of that set.

   So, by the definition of an open set:

   > U is *open* if for each $x \in U$ there is $\varepsilon > 0$ such that $y \in U$ whenever $d(y, x) < \varepsilon$,

   the set E_K° is open in E.

3. *Let E be a normed space. Let $K > 0$. Prove that $\overline{E_K^\circ} = E_K$.*

   We must show that E_K is precisely the set of vectors which are the limits of sequences of points of E_K°. But if $x \in E_K$ let $x_n = \frac{n-1}{n}x$ for $n \in \mathbb{N}$. Then $\|x_n\| = \frac{n-1}{n}\|x\| \leqslant \frac{n-1}{n}K < K$ so $x_n \in E_K^\circ$ and $x_n \to x$ by the continuity of scalar multiplication.

   Conversely, let (x_n) be a sequence in E_K° converging to a point $x \in E$. Then $\|x_n\| < K$ and, by the continuity of the norm $\|x_n\| \to \|x\|$. Therefore $\|x\| \leqslant K$, and so $x \in E_K$ as required.

PMA445 Functional Analysis 2004
Question Sheet 3

To be handed in on Tuesday 9 March.

1. Let $\ell^1 = \ell^1(\mathbb{N})$ denote the set of all infinite sequences $\boldsymbol{x} = (x_1, x_2, \ldots, x_n, \ldots)$ of complex numbers for which the series $\sum_{n=1}^{\infty} |x_n|$ converges. Prove that if we define

$$\|\boldsymbol{x}\| = \sum_{n=1}^{\infty} |x_n|,$$

then:
(i) ℓ^1 is a normed space;
(ii) ℓ^1 is a Banach space (i.e is is also complete).

[Hint for (ii): adapt (in fact simplify) the proof we had in the lectures for ℓ^2.]

2. Let E and f be two normed spaces. Then $E \times F = \{(x, y) : x \in E \text{ and } y \in F\}$ is a vector space. For $(x, y) \in E \times F$ define

$$\|(x, y)\| = \|x\| + \|y\|.$$

(i) Show that $E \times F$ is a normed space.

(ii) Show that a sequence (x_n, y_n) converges to (x, y) in $E \times F$ *if and only if* (x_n) converges to x in E and (y_n) converges to y in F.

(iii) Use (ii), with $F = E$ and then with $F = \mathbb{C}$, to show that Theorem 6.4 parts 2 and 3 can be interpreted as saying that the map $(x, y) \mapsto x + y$ of $E \times E$ to E, and the map $(x, \lambda) \mapsto \lambda x$ of $E \times \mathbb{C}$ to E are both continuous.

(iv) $*$ Prove that if E and F are complete then so is $E \times F$. [Hint: Take a Cauchy sequence in (x_n, y_n) in $E \times F$, and show that (x_n) is Cauchy in E; at the end use (ii).]

(v) Prove that if $E \times F$ is complete then so is E. [Hint: if (x_n) is Cauchy in E then $(x_n, 0)$ is easily shown to be Cauchy in $E \times F$.]

PMA445 Functional Analysis 2004
Question Sheet 3: Solutions

1. *Let $\ell^1 = \ell^1(\mathbb{N})$ denote the set of all infinite sequences $\boldsymbol{x} = (x_1, x_2, \ldots, x_n, \ldots)$ of complex numbers for which the series $\sum_{n=1}^{\infty} |x_n|$ converges. Prove that if we define*

$$\|\boldsymbol{x}\| = \sum_{n=1}^{\infty} |x_n|,$$

then:
(i) ℓ^1 is a normed space;
(ii) ℓ^1 is a Banach space (i.e is also complete).

(i) We check simultaneously that ℓ^1 is closed under addition and scalar multiplication and that the axioms for a norm are satisfied.

(a) For $\boldsymbol{x}, \boldsymbol{y} \in \ell^1$,

$$\begin{aligned} \|\boldsymbol{x} + \boldsymbol{y}\| &= \sum_{n=1}^{\infty} |x_n + y_n| && (1) \\ &\leqslant \sum_{n=1}^{\infty} |x_n| + |y_n| && (2) \\ &= \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| && (3) \\ &= \|\boldsymbol{x}\| + \|\boldsymbol{y}\|, \end{aligned}$$

where the convergence of the sums (3) ensures the convergence of the sum (2) and hence of the sum (1). Thus $\boldsymbol{x} + \boldsymbol{y} \in \ell^1$ and the triangle inequality holds.

(b) For $\boldsymbol{x} \in \ell^1$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned} \|\lambda \boldsymbol{x}\| &= \sum_{n=1}^{\infty} |\lambda x_n| && (4) \\ &\leqslant \sum_{n=1}^{\infty} |\lambda|\,|x_n| && (5) \\ &= |\lambda| \sum_{n=1}^{\infty} |x_n| && (6) \\ &= |\lambda|\,\|\boldsymbol{x}\|, \end{aligned}$$

where the convergence of the sum in (6) ensures the convergence of the sum (5) and hence of the sum (4). Thus $\lambda \boldsymbol{x} \in \ell^1$ and the second axiom for a norm holds.

(c) For $\boldsymbol{x} \in \ell^1$ we have

$$\begin{aligned} \|\boldsymbol{x}\| = 0 \quad &\Longleftrightarrow \quad \sum_{n=1}^{\infty} |x_n| = 0 \\ &\Longleftrightarrow \quad |x_n| = 0 \text{ for all } n \\ &\Longleftrightarrow \quad x_n = 0 \text{ for all } n \\ &\Longleftrightarrow \quad \boldsymbol{x} = 0. \end{aligned}$$

(ii) We show that ℓ^1 is complete, mimicking the proof for ℓ^2. Let $(\boldsymbol{x}^{(\ell)})$ be a Cauchy sequence in ℓ^1. Then given $\varepsilon > 0$ we can find $L \in \mathbb{N}$ such that

$$\left\| \boldsymbol{x}^{(k)} - \boldsymbol{x}^{(\ell)} \right\| = \sum_{n=1}^{\infty} \left| x_n^{(k)} - x_n^{(\ell)} \right| < \varepsilon \qquad (k, \ell \geqslant L).$$

Therefore

$$\sum_{n=1}^{\infty} \left| x_n^{(k)} - x_n^{(\ell)} \right| < \varepsilon \qquad (k, \ell \geqslant L).$$

Therefore, for all $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \left| x_n^{(k)} - x_n^{(\ell)} \right| < \varepsilon \qquad (k, \ell \geqslant L). \tag{7}$$

The remainder consists of three steps which are standard for proofs of completeness.

(A) Construct $\boldsymbol{x}$. Fixing n we deduce from (7) that

$$\left| x_n^{(k)} - x_n^{(\ell)} \right| < \varepsilon \qquad (k, \ell \geqslant L).$$

Therefore the sequence of complex numbers $(x_n^{(k)})$ is Cauchy in $\mathbb{C}$, and so converges. So let

$$x_n = \lim_{k \to \infty} x_n^{(k)} \qquad (n \in \mathbb{N}).$$

This gives us a sequence $\boldsymbol{x} = (x_n)$ of complex numbers.

(B) Show that $\boldsymbol{x} \in \ell$. To do this we use the fact that the sequence $\left(\boldsymbol{x}^{(k)}\right)$, being Cauchy, is bounded, say $\left\|\boldsymbol{x}^{(k)}\right\| \leqslant M$ for all k. So

$$\sum_{n=1}^{\infty} \left| x_n^{(k)} \right| \leqslant M \qquad \text{for all } k,$$

Therefore

$$\sum_{n=1}^{N} \left| x_n^{(k)} \right| \leqslant M \qquad \text{for all } k, N.$$

Therefore, letting $k \to \infty$,

$$\sum_{n=1}^{N} |x_n| \leqslant M \qquad \text{for all } N,$$

and letting $N \to \infty$, we get

$$\sum_{n=1}^{\infty} |x_n| \leqslant M < \infty.$$

So we have shown that $\boldsymbol{x} \in \ell^1$.

(C) Show that $\left(\boldsymbol{x}^{(k)}\right)$ converges to $\boldsymbol{x}$. We return to (7), fix N, fix $k \geqslant L$, and let $\ell \to \infty$. Then

$$\sum_{n=1}^{N} \left| x_n^{(k)} - x_n \right| = \lim_{\ell\to\infty} \sum_{n=1}^{N} \left| x_n^{(k)} - x_n^{(\ell)} \right| \leqslant \varepsilon \qquad (N \in \mathbb{N},\ k \geqslant L).$$

Letting $N \to \infty$ we get

$$\sum_{n=1}^{\infty} \left| x_n^{(k)} - x_n \right| \leqslant \varepsilon \qquad (k \geqslant L).$$

So

$$\left\| \boldsymbol{x}^{(k)} - \boldsymbol{x} \right\| \leqslant \varepsilon \qquad (k \geqslant L).$$

Therefore, since ε is arbitrary, $\boldsymbol{x}^{(k)} \to \boldsymbol{x}$. So $\left(\boldsymbol{x}^{(k)}\right)$ converges, and we have shown that ℓ^1 is complete, i.e. is a Hilbert space.

2. *Let E and F be two normed spaces. Then $E \times F = \{(x, y) : x \in E \text{ and } y \in F\}$ is a vector space. For $(x, y) \in E \times F$ define*

$$\|(x, y)\| = \|x\| + \|y\|.$$

(i) *Show that $E \times F$ is a normed space.*

We take the fact that $E \times F$ is a vector space as known and check the that axioms for the norm on $E \times F$ follow from the same axioms for the norms on E and on F.

(a) For $(x_1, y_1), (x_2, y_2) \in E \times F$,

$$\begin{aligned} \|(x_1, y_1) + (x_2, y_2)\| &= \|(x_1 + x_2, y_1 + y_2)\| \\ &= \|x_1 + x_2\|_E + \|y_1 + y_2\|_F \\ &\leqslant \|x_1\|_E + \|x_2\|_E + \|y_1\|_F + \|y_2\|_F \\ &= \|(x_1, y_1)\| + \|(x_2, y_2)\|. \end{aligned}$$

(b) For $(x, y) \in E \times F$ and $\lambda \in \mathbb{C}$,

$$\|\lambda(x, y)\| = \|(\lambda x, \lambda y)\| = \|\lambda x\|_E + \|\lambda y\|_F \leqslant |\lambda|\,\|x\|_E + |\lambda|\,\|y\|_F = |\lambda|\,\|(x, y)\|.$$

(c) For $(x, y) \in E \times F$,

$$\|(x, y)\| = 0 \iff \|x\|_E + \|y\|_F = 0 \iff \|x\|_E = 0 = \|y\|_F \iff (x, y) = (0, 0).$$

(ii) *Show that a sequence (x_n, y_n) converges to (x, y) in $E \times F$ if and only if (x_n) converges to x in E and (y_n) converges to y in F.*

$$\begin{aligned} (x_n, y_n) \to (x, y) &\iff \|(x_n, y_n) - (x, y)\| \to 0 \\ &\iff \|(x_n - x, y_n - y)\| \to 0 \\ &\iff \|x_n - x\|_E + \|y_n - y\|_F \to 0 \\ &\iff \|x_n - x\|_E \to 0 \text{ and } \|y_n - y\|_F \to 0 \\ &\iff x_n \to x \text{ in } E \text{ and } y_n \to y \text{ in } F. \end{aligned}$$

(iii) *Use (ii), with $F = E$ and then with $F = \mathbb{C}$, to show that Theorem 6.4 parts 2 and 3 can be interpreted as saying that the map $(x, y) \mapsto x + y$ of $E \times E$ to E, and the map $(x, \lambda) \mapsto \lambda x$ of $E \times \mathbb{C}$ to E are both continuous.*

Saying that the maps

$$(x, y) \mapsto x + y \ : \ E \times E \to E,$$
$$(x, \lambda) \mapsto \lambda x \ : \ E \times \mathbb{C} \to E$$

are each continuous is just saying that

$$(x_n, y_n) \to (x, y) \quad \Rightarrow \quad x_n + y_n \to x + y,$$
$$(x_n, \lambda_n) \to (x, \lambda) \quad \Rightarrow \quad \lambda_n x_n \to \lambda x,$$

respectively. Using (ii) turns these into:

$$x_n \to x \ \ \& \ \ y_n \to y \quad \Rightarrow \quad x_n + y_n \to x + y,$$
$$x_n \to x \ \ \& \ \ \lambda_n \to \lambda \quad \Rightarrow \quad \lambda_n x_n \to \lambda x,$$

respectively, and this is the content of Theorem 6.4 parts 2 and 3

(iv) *Prove that if E and F are complete then so is $E \times F$.*

Suppose E and F are complete. We show that $E \times F$ is complete. Let (x_n, y_n) be a Cauchy sequence in $E \times F$. Then, for all $\varepsilon > 0$ there exists N such that for all $p, q \geqslant N$

$$\|(x_p, y_p) - (x_q, y_q)\| < \varepsilon.$$

Therefore

$$\|x_p - x_q\| \leqslant \|x_p - x_q\| + \|y_p - y_q\| = \|(x_p - x_q, y_p - y_q)\| = \|(x_p, y_p) - (x_q, y_q)\| < \varepsilon.$$

Hence (x_n) is Cauchy in E; similarly, (y_n) is Cauchy in F. Since E and F are assumed complete, we have $x_n \to x$ and $y_n \to y$ for some $x \in E$, $y \in F$. By (ii), $(x_n, y_n) \to (x, y)$, as required.

(v) *Prove that if $E \times F$ is complete then so is E.*

Now suppose that $E \times F$ is complete. We show that E is complete. Let (x_n) be a Cauchy sequence in E. Then for all $\varepsilon > 0$ there exists N such that for all $p, q \geqslant N$

$$\|x_p - x_q\| < \varepsilon.$$

Therefore

$$\|(x_p, 0) - (x_q, 0)\| = \|(x_p - x_q, 0)\| = \|x_p - x_q\| < \varepsilon.$$

Thus $(x_n, 0)$ is Cauchy in $E \times F$. Since $E \times F$ is assumed complete, we have $(x_n, 0) \to (x, y)$ for some $x \in E$, $y \in F$. It follows from (ii) that $x_n \to x$, as required.

Notice that the same argument, or an observation of 'symmetry', shows that $E \times F$ complete implies F complete. We may sum up the main results of this question by saying that '$E \times F$ is a normed space which is complete if and only if both E and F are complete.

PMA445 Functional Analysis 2004
Question Sheet 4

Not to be handed in. Solutions will be distributed on Tuesday 16 March.

1. The norm of a linear functional f on a normed space E is defined by

$$\|f\| = \sup\{|f(\boldsymbol{x})| : \ \boldsymbol{x} \in E, \|x\| \leqslant 1\}.$$

The purpose of this exercise is to show that the sup is not necessarily attained (even for a Banach space E).

Let $E = c_0$, the space of sequences convergent to zero with the usual supremum norm

$$\|\boldsymbol{x}\| = \sup\{|x_n| : n = 1, 2, 3, \ldots\} \qquad (\boldsymbol{x} \in c_0).$$

Let f be the linear functional on c_0 defined by

$$f(\boldsymbol{x}) = \sum_{n=1}^{\infty} 2^{-n} x_n \qquad (\boldsymbol{x} \in c_0).$$

Show that $\|f\| = 1$ but that there is no $\boldsymbol{x} \in c_0$ with $\|\boldsymbol{x}\| \leqslant 1$ and $|f(\boldsymbol{x})| = 1$.

2. Recall that $\ell^1 = \ell^1(\mathbb{N})$ is the set of all infinite sequences $\boldsymbol{x} = (x_1, x_2, \ldots x_n, \ldots)$ of complex numbers for which the series $\sum_{n=1}^{\infty} |x_n|$ converges, with the norm given by

$$\|\boldsymbol{x}\| = \sum_{n=1}^{\infty} |x_n|.$$

Define a linear map $L : \ell^1 \to \ell^1$ by

$$(L\boldsymbol{x})_n = x_{n+1} \qquad (\boldsymbol{x} \in \ell^1; \ n \in \mathbb{N}).$$

Show that L is bounded with $\|L\| \leqslant 1$. But choosing a suitable $\boldsymbol{x}$, **in** ℓ^1, for which $\|L\boldsymbol{x}\| = \|\boldsymbol{x}\|$ show that $\|L\| = 1$. [This map is called the *left shift* for obvious reasons.]

3. Let E, F be normed spaces. Let f be a continuous linear functional on E. Let $y \in F$. Define $T : E \to F$ by

$$Tx = f(x)y \qquad (x \in E).$$

Show that T is linear and bounded and find $\|T\|$ in terms of $\|f\|$ and $\|y\|$. [All linear mapping of *rank one* — i.e. mappings whose range is one-dimensional — are of this form.]

4. Prove that if $E \neq \{0\}$ and $\mathcal{B}(E, F)$ is complete then F is complete. This completes the proof of Theorem 7.5. [Hint: You may assume, as we prove in the chapter on dual spaces, there actually is a bounded linear functional f on E of norm 1. Let (y_n) be a Cauchy sequence in F. Construct T_n as in Question 3. Show that (T_n) is Cauchy in $\mathcal{B}(E, F)$.]

PMA445 Functional Analysis 2004
Question Sheet 4: Solutions

1. *Let $E = c_0$, the space of sequences convergent to zero with the usual supremum norm*

$$\|\boldsymbol{x}\| = \sup\{|x_n| : n = 1, 2, 3, \ldots\} \qquad (\boldsymbol{x} \in c_0).$$

Let f be the linear functional on c_0 defined by

$$f(\boldsymbol{x}) = \sum_{n=1}^{\infty} 2^{-n} x_n \qquad (\boldsymbol{x} \in c_0).$$

Show that $\|f\| = 1$ but that there is no $\boldsymbol{x} \in c_0$ with $\|\boldsymbol{x}\| \leqslant 1$ and $|f(\boldsymbol{x})| = 1$.

If $\|\boldsymbol{x}\| \leqslant 1$, then $|x_n| \leqslant 1$ for all n, so

$$\begin{aligned} |f(\boldsymbol{x})| &= \left|\sum_{n=1}^{\infty} 2^{-n} x_n\right| \\ &\leqslant \sum_{n=1}^{\infty} 2^{-n} |x_n| \\ &\leqslant \sum_{n=1}^{\infty} 2^{-n} \qquad (1) \\ &= 1. \end{aligned}$$

Therefore $\|f\| \leqslant 1$.

On the other hand, if $\boldsymbol{e}_k = (1, 1, 1, \ldots, 1, 0, 0, 0, \ldots)$, with k ones, then $\|\boldsymbol{e}_k\| = 1$ and

$$f(\boldsymbol{e}_k) = \sum_{n=1}^{k} 2^{-n} = 1 - 2^{-k}.$$

Therefore $\|f\| \geqslant 1 - 2^{-k}$ for all k, so $\|f\| = 1$.

However, in order to have $\|\boldsymbol{x}\| \leqslant 1$ and $|f(\boldsymbol{x})| = 1$ we should have to have equality in (1) above. This means we should have to have $|x_n| = 1$ for all n, which is incompatible with the requirement $\boldsymbol{x} \in c_0$, i.e. $x_n \to 0$ as $n \to \infty$.

2. *Define a linear map $L : \ell^1 \to \ell^1$ by*

$$(L\boldsymbol{x})_n = x_{n+1} \qquad (\boldsymbol{x} \in \ell^1;\ n \in \mathbb{N}).$$

Show that L is bounded with $\|L\| \leqslant 1$. But choosing a suitable $\boldsymbol{x}$, **in** *ℓ^1, for which $\|L\boldsymbol{x}\| = \|\boldsymbol{x}\|$ show that $\|L\| = 1$.*

We have

$$\|L\boldsymbol{x}\| = \sum_{n=1}^{\infty} |x_{n+1}| = \sum_{n=2}^{\infty} |x_n| \leqslant \sum_{n=1}^{\infty} |x_n| = \|\boldsymbol{x}\| \qquad (\boldsymbol{x} \in \ell^1),$$

so $\|L\| \leqslant 1$.

If we choose any non-zero $\boldsymbol{x}$ with $x_1 = 0$, e.g. $\boldsymbol{x} = (0, 1, 0, 0, \ldots)$, then the above inequality becomes equality, showing that $\|L\| = 1$. [Note: $(0, 1, 1, 1, \ldots)$ would not be a valid choice as it is not in ℓ^1.]

3. *Let E, F be normed spaces. Let f be a continuous linear functional on E. Let $y \in F$. Define $T : E \to F$ by*
$$Tx = f(x)y \qquad (x \in E).$$
Show that T is linear and bounded and find $\|T\|$ in terms of $\|f\|$ and $\|y\|$.

To show that T is linear, let $x_1, x_2 \in E$ and $\lambda_1, \lambda_2 \in \mathbb{C}$; then

$$\begin{aligned} T(\lambda_1 x_1 + \lambda_2 x_2) &= f(\lambda_1 x_1 + \lambda_2 x_2)y \\ &= [\lambda_1 f(x_1) + \lambda_2 f(x_2)]\, y, \text{ since } f \text{ is linear,} \\ &= \lambda_1 f(x_1)y + \lambda_2 f(x_2)y \\ &= \lambda_1 T(x_1) + \lambda_2 T(x_2). \end{aligned}$$

Since f is continuous,
$$|f(x)| \leqslant \|f\|\,\|x\| \qquad (x \in E).$$
Then,
$$\|Tx\| = |f(x)|\,\|y\| \leqslant \|f\|\,\|x\|\,\|y\| = (\|f\|\,\|y\|)\|x\| \qquad (x \in E),$$
so T is bounded and

$$\begin{aligned} \|T\| &= \min\{k \in \mathbb{R}^+ : \|Tx\| \leqslant k\|x\|\} \\ &= \min\{k \in \mathbb{R}^+ : |f(x)|\,\|y\| \leqslant k\|x\|\} \\ &= \|y\| \min\{k \in \mathbb{R}^+ : |f(x)| \leqslant k\|x\|\} \\ &= \|y\|\,\|f\|. \end{aligned}$$

4. *Prove that if $E \neq \{0\}$ and $\mathcal{B}(E, F)$ is complete then F is complete.*

Let (y_n) be Cauchy in F. Let f be a bounded linear functional on E with $\|f\| = 1$. So, as in Question 3, let
$$T_n x = f(x)y_n \qquad (x \in E).$$
Then $(T_n - T_m)(x) = f(x)[y_n - y_m]$. So applying Question 3 to $T = T_n - T_m$ we get
$$\|T_n - T_m\| = \|f\|\,\|y_n - y_m\| = \|y_n - y_m\|.$$
Therefore, the assumption that (y_n) is Cauchy implies that (T_n) is Cauchy in $\mathcal{B}(E, F)$. Then (T_n) converges, to $T \in \mathcal{B}(E, F)$, say. Choose x with $f(x) = 1$, and let $y = Tx$. Then $y_n = f(x)y_n = T_n x \to Tx = y$, so (y_n) converges, as required.

PMA445 Functional Analysis 2004
Question Sheet 5

To be handed in on Tuesday 23 March.

1. Let E, F, G be normed spaces. Let $S \in \mathcal{B}(E,F)$ and $T \in \mathcal{B}(F,G)$. Clearly $T \circ S$ is linear (where $T \circ Sx = T(Sx)$ for $x \in E$). Show that $T \circ S$ is bounded, by showing that

$$\|T \circ S\| \leqslant \|T\|\,\|S\|.$$

Construct an example to show that sometimes one can have

$$\|T \circ S\| < \|T\|\,\|S\|.$$

[**Hint:** to do this it will suffice to find two non-zero linear maps S, T for which $T \circ S = 0$. (Why?) Try taking $E = F = G = \mathbb{R}^2$ – note that, their norms will be immaterial.]

2. Let E, F be normed spaces, and T a bounded linear mapping of E into F. Let $E \times F$ be a normed space for the norm

$$\|(x,y)\| = \max\{\|x\|, \|y\|\} \qquad \text{for } x \in E,\ y \in F.$$

Let $G \subseteq E \times F$ be the graph of T, i.e.

$$G = \{(x, Tx) : x \in E\}.$$

Show that G is a closed set in $E \times F$.

[**Hints:** as in the easy part of Exercise Sheet 3, Question 2(ii), show that if $(x_n, y_n) \to (x,y)$ then $x_n \to x$ and $y_n \to y$; note that $(x,y) \in G$ means exactly that $y = Tx$.]

3. Prove that the dual of ℓ^1 is ℓ^∞. Specifically, prove the following:

   (i) every $\boldsymbol{x} = (x_1, x_2, \ldots) \in \ell^\infty$ defines a continuous linear functional $f_{\boldsymbol{x}}$ on ℓ^1 by

$$f_{\boldsymbol{x}}(\boldsymbol{y}) = \sum_{i=1}^{\infty} x_i y_i \qquad (\boldsymbol{y} = (y_1, y_2, \ldots) \in \ell^1);$$

   (ii) every continuous linear functional on ℓ^1 is of this form;

   (iii) $\|f_{\boldsymbol{x}}\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \ell^\infty$, (where $\|f_{\boldsymbol{x}}\|$ is the norm of the linear functional and $\|\boldsymbol{x}\|$ is the ℓ^∞-norm of $\boldsymbol{x}$).

PMA445 Functional Analysis 2004
Question Sheet 5: Solutions

1. *Let E, F, G be normed spaces. Let $S \in \mathcal{B}(E, F)$ and $T \in \mathcal{B}(F, G)$. Clearly $T \circ S$ is linear (where $T \circ Sx = T(Sx)$ for $x \in E$). Show that $T \circ S$ is bounded, by showing that*

$$\|T \circ S\| \leqslant \|T\| \, \|S\|.$$

Construct an example to show that sometimes one can have

$$\|T \circ S\| < \|T\| \, \|S\|.$$

By the definition of the operator norm, for all $x \in E$,

$$\begin{aligned} \|T \circ Sx\| &= \|T(Sx)\| \\ &\leqslant \|T\| \, \|Sx\| \\ &\leqslant \|T\|(\|S\| \, \|x\|) \\ &\leqslant (\|T\| \, \|S\|)\|x\|, \end{aligned}$$

so $T \circ S$ is bounded with $\|T \circ S\| \leqslant \|T\| \, \|S\|$.

But now let say $F = \mathbb{R}^2$ with, say, the norm

$$\|(x, y)\| = \max\{|x|, |y|\}.$$

Let S, T mapping F into F be defined by

$$T(x, y) = (x, 0) \qquad \text{and} \qquad S(x, y) = (0, y).$$

Then clearly $T \circ S = 0$ so that $\|T \circ S\| = 0$. But since $T \neq 0$ and $S \neq 0$ then $\|T\| \, \|S\| > 0.0 = 0$. (In fact $\|T\| = \|S\| = 1$, for $\|T\| \leqslant 1$ is clear since $\|T(x, y)\| = \|(x, 0)\| = \|x\| \leqslant \|(x, y)\|$, and $\|T(1, 0)\| = \|(1, 0)\|$.)

2. *Let E, F be normed spaces, and T a bounded linear mapping of E into F. Let $E \times F$ be a normed space for the norm*

$$\|(x, y)\| = \max\{\|x\|, \|y\|\} \qquad \text{for } x \in E,\ y \in F.$$

Let $G \subseteq E \times F$ be the graph of T, i.e.

$$G = \{(x, Tx) : x \in E\}.$$

Show that G is a closed set in $E \times F$.

Let (x_n, y_n) be a sequence in G converging to (x, y). Since $(x_n, y_n) \in G$ then $y_n = Tx_n$. Then $\|x_n - x\| \leqslant \|(x_n - x, y_n - y)\| = \|(x_n, y_n) - (x, y)\| \to 0$, so that $\|x_n - x\| \to 0$, i.e. $x_n \to x$; similarly $y_n \to y$. Since T is continuous Tx_n converges to Tx. So $y_n = Tx_n$ converges to both y and Tx. Therefore $y = Tx$ so that $(x, y) = (x, Tx)$ is in G. So G is closed.

3. *Prove that the dual of ℓ^1 is ℓ^∞. Specifically, prove the following:*

(i) every $\boldsymbol{x} = (x_1, x_2, \ldots) \in \ell^\infty$ *defines a continuous linear functional* $f_{\boldsymbol{x}}$ *on* ℓ^1 *by*

$$f_{\boldsymbol{x}}(\boldsymbol{y}) = \sum_{i=1}^{\infty} x_i y_i \qquad (\boldsymbol{y} = (y_1, y_2, \ldots) \in \ell^1); \tag{1}$$

(ii) every continuous linear functional on ℓ^1 *is of this form;*

(iii) $\|f_{\boldsymbol{x}}\| = \|\boldsymbol{x}\|$ *for all* $\boldsymbol{x} \in \ell^\infty$*, (where* $\|f_{\boldsymbol{x}}\|$ *is the norm of the linear functional and* $\|\boldsymbol{x}\|$ *is the* ℓ^∞*-norm of* $\boldsymbol{x}$*).*

(i) We must first observe that the sum

$$\sum_{i=1}^{\infty} x_i y_i$$

is (absolutely) convergent for every $\boldsymbol{x} \in \ell^\infty$ and $\boldsymbol{y} \in \ell^1$, since

$$\sum_{i=1}^{\infty} |x_i y_i| \leqslant \|\boldsymbol{x}\|_\infty \sum_{i=1}^{\infty} |y_i| < \infty.$$

Thus (1) does define $f_{\boldsymbol{x}}(\boldsymbol{y})$. The proof that $f_{\boldsymbol{x}}$ is linear is straightforward and is omitted. We prove that $f_{\boldsymbol{x}}$ is continuous. This follows from essentially the same inequalities:

$$|f_{\boldsymbol{x}}(\boldsymbol{y})| \leqslant \sum_{i=1}^{\infty} |x_i y_i| \leqslant \|\boldsymbol{x}\|_\infty \, \|\boldsymbol{y}\|_1.$$

Thus $f_{\boldsymbol{x}} \in (\ell^1)'$, and we also have $\|f_{\boldsymbol{x}}\| \leqslant \|x\|_\infty$.

(ii) We must show that an arbitrary $f \in (\ell^1)'$ is of the form $f_{\boldsymbol{x}}$ for some $\boldsymbol{x} \in \ell^\infty$. Define $\boldsymbol{e}_n \in \ell^1$ $(n = 1, 2, 3, \ldots)$ by

$$(\boldsymbol{e}_n)_i = \begin{cases} 1 & (i = n) \\ 0 & (i \neq n). \end{cases}$$

Observation: (not part of the proof) if we had $f = f_{\boldsymbol{x}}$, then we should have $f(\boldsymbol{e}_n) = x_n$. Returning to the proof, we define a sequence $\boldsymbol{x} = (x_1, x_2, \ldots)$, using the given functional f, by

$$x_n = f(\boldsymbol{e}_n) \qquad (n = 1, 2, 3, \ldots).$$

Then

$$|x_n| = |f(\boldsymbol{e}_n)| \leqslant \|f\| \, \|\boldsymbol{e}_n\| = \|f\|,$$

so $\boldsymbol{x} \in \ell^\infty$. Indeed,

$$\|\boldsymbol{x}\|_1 = \sup_n |x_n| \leqslant \|f\|.$$

We have defined $\boldsymbol{x}$ so that $f_{\boldsymbol{x}}(\boldsymbol{y}) = f(\boldsymbol{y})$ for $\boldsymbol{y} \in \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots\}$. Since both f and $f_{\boldsymbol{x}}$ are linear, it follows that $f_{\boldsymbol{x}}(\boldsymbol{y}) = f(\boldsymbol{y})$ for all $\boldsymbol{y} \in \operatorname{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots\} = c_{00}$. Then, since both f and $f_{\boldsymbol{x}}$ are continuous, it follows that $f_{\boldsymbol{x}}(\boldsymbol{y}) = f(\boldsymbol{y})$ for all $\boldsymbol{y} \in \overline{c_{00}} = \ell^1$: i.e. $f = f_{\boldsymbol{x}}$.

(iii) We have already shown that $\|f_{\boldsymbol{x}}\| \leqslant \|x\|_\infty$ and $\|x\|_\infty \leqslant \|f\|$, so $\|f_{\boldsymbol{x}}\| = \|x\|_\infty$.

PMA445 Functional Analysis 2004
Question Sheet 6

To be handed in on Thursday 29 April.

1. Give the proofs of the linearity checks which were dismissed as obvious in the proof of Theorem 11.3. (Just to check that they are obvious!) Specifically, prove (a) that $\nu(x)$ is a linear functional on E', for each $x \in E$, and (b) the mapping $\nu : E \to E''$ is linear.

2. Let E, F be normed spaces.

   (a) Show that for every $T \in \mathcal{B}(E, F)$ there is a $T^* \in \mathcal{B}(F', E')$ given by
   $$(T^*g)(x) = g(Tx) \qquad (g \in F',\ x \in E),$$
   and $\|T^*\| \leqslant \|T\|$. (If you have done Question 1, you may dismiss the linearity proofs here as obvious.)

   (b) Show that $T^{**} \in \mathcal{B}(E'', F'')$ is an extension of the map $\nu T \nu^{-1} : \nu(E) \to \nu(F)$. [If you think of E as identified with the subspace $\nu(E)$ of E'', this just says that T^{**} is an extension on T.]

   (c) Deduce that $\|T^{**}\| \geqslant \|T\|$. Hence, or otherwise prove that $\|T^*\| = \|T\|$.

   (d) If $S \in \mathcal{B}(E, F)$ and $T \in \mathcal{B}(F, G)$, what can you say about $(TS)^*$?

PMA445 Functional Analysis 2004
Question Sheet 6: Solutions

1. *Give the proofs of the linearity checks which were dismissed as obvious in the proof of Theorem 11.3. Specifically, prove (a) that $\nu(x)$ is a linear functional on E', for each $x \in E$, and (b) the mapping $\nu : E \to E''$ is linear.*

   (a) Given $x \in E$, for all $f, g \in E'$ and $\lambda, \mu \in \mathbb{C}$, we have

$$\begin{aligned}(\nu(x))(\lambda f + \mu g) &= (\lambda f + \mu g)(x) \\ &= \lambda f(x) + \mu g(x) \\ &= \lambda(\nu(x))(f) + \mu(\nu(x))(g).\end{aligned}$$

   Therefore $\nu(x)$ is linear.

   (b) For all $x, y \in E$ and $\lambda, \mu \in \mathbb{C}$, and all $f \in E'$,

$$\begin{aligned}(\nu(\lambda x + \mu y))(f) &= f(\lambda x + \mu y) \\ &= \lambda f(x) + \mu f(y), \text{ by the linearity of } f, \\ &= \lambda(\nu(x))(f) + \mu(\nu(y))(f) \\ &= (\lambda\nu(x) + \mu\nu(y))(f).\end{aligned}$$

   Therefore

$$\nu(\lambda x + \mu y) = \lambda\nu(x) + \mu\nu(y).$$

   Thus ν is linear.

2. *Let E, F be normed spaces.*

   (a) Show that for every $T \in \mathcal{B}(E, F)$ there is a $T^* \in \mathcal{B}(F', E')$ given by

$$(T^*g)(x) = g(Tx) \qquad (g \in F',\ x \in E), \tag{1}$$

   and $\|T^*\| \leqslant \|T\|$.

   (b) Show that $T^{**} \in \mathcal{B}(E'', F'')$ is an extension of the map $\nu T \nu^{-1} : \nu(E) \to \nu(F)$.

   (c) Deduce that $\|T^{**}\| \geqslant \|T\|$. Hence, or otherwise prove that $\|T^*\| = \|T\|$.

   (d) If $S \in \mathcal{B}(E, F)$ and $T \in \mathcal{B}(F, G)$, what can you say about $(TS)^*$?

   (a) We first prove that, for each $g \in F'$, the mapping T^*g defined by (1) is a continuous linear functional on E. Linearity is obvious. To prove continuity:

$$|(T^*g)(x)| = |g(Tx)| \leqslant \|g\|\,\|Tx\| \leqslant \|g\|\,\|T\|\,\|x\|.$$

   Thus $T^*g \in E'$ with

$$\|T^*g\| \leqslant \|g\|\,\|T\|. \tag{2}$$

   Next we observe that the mapping $T^* : F' \to E'$ is linear (obvious), and continuous (by (2)) with $\|T^*\| \leqslant \|T\|$.

(b) If $x \in E$, then $T^{**}(\nu(x)) \in F''$, so we want to know its effect on a typical $g \in F'$:

$$\begin{aligned}
\left(T^{**}(\nu(x))\right)(g) &= (\nu(x))(T^*g) \\
&= (T^*g)(x) \\
&= g(Tx) \\
&= (\nu(Tx))(g) \\
&= \left(\nu T\nu^{-1}(\nu(x))\right)(g).
\end{aligned}$$

Therefore $T^{**} = \nu T \nu^{-1}$ on $\nu(E)$.

(c) Since ν is isometric, it follows that the norm T^{**} on $\nu(E)$ is equal to $\|T\|$, so the norm of T^{**} on E'' is at least $\|T\|$. Therefore

$$\|T\| \leqslant \|T^{**}\| \leqslant \|T^*\| \leqslant \|T\|,$$

so they must all be equal: $\|T^*\| = \|T\|$.

(d) Since $TS \in \mathcal{B}(E, G)$, we have $(TS)^* \in \mathcal{B}(G', E')$, so we must look at its effect on $h \in G'$. Now $(TS)^*(h) \in E'$, so we look at its effect on $x \in E$:

$$((TS)^*(h))(x) = h(TSx) = (T^*h)(Sx) = (S^*(T^*h))(x).$$

This holds for all $x \in E$, so

$$(TS)^*(h) = S^*(T^*h) = (S^*T^*)(h).$$

This holds for all $h \in G'$, so $(TS)^* = S^*T^*$.

PMA445 Functional Analysis 2004
Question Sheet 7

Last question sheet. Not to be handed in.

1. Show that we cannot add to the conclusions of Corollary 12.6 that $f_n \to f$ in E'. (Hint: let E be a suitable space of sequences — you could choose c_0 or ℓ^p for some $p \in [1, \infty)$ — and let f_n be the linear functional associated with an element $(0, 0, \ldots, 0, 1, 0, \ldots)$ in E'.)

2. Let $C[0,1]$ denote the space of all continuous complex-valued functions on $[0,1]$ with the sup norm. Let $P_k \subseteq C[0,1]$ be the subspace of all polynomials of degree at most k. Let $t_0, t_1, \ldots, t_k$ be distinct points in $[0,1]$. Show that if $T : C[0,1] \to P_k$ is a linear mapping such that
$$(Tx)(t_i) = x(t_i) \qquad (x \in C[0,1],\ 0 \leqslant i \leqslant k),$$
then T is continuous.

Non-examinable proofs

The following pieces of text will not be examined as bookwork.

1. All of Section 8, the Axiom of Choice.

2. The proof of Theorem (9.3), the Hahn–Banach Theorem for real vector spaces.

3. The proof of Theorem (12.3), the Uniform Boundedness Theorem.

4. The proof of Theorem (13.3), the Open Mapping Theorem.

Note that you will be expected to know the statements of Theorems (9.3), (12.3) and (13.3) and be able to apply these results.

In general you can expect the examination paper to test your knowledge of basic **definitions**, without which you would not know what you were talking about, then the statements of **key theorems**, and then the ability to solve problems and reconstruct proofs encountered in lectures. Past papers should be a good guide to the type of questions to be set, though note that the June 2004 paper will be in a **new format with a compulsory question 1**.

PMA445 Functional Analysis 2004
Question Sheet 7: Solutions

1. *Show that we cannot add to the conclusions of Corollary 12.6 that $f_n \to f$ in E'.*

   Let $E = c_0$, so $E' = \ell^1$ and let $f_n \in E'$ be defined by

   $$f_n(\boldsymbol{x}) = x_n \qquad (\boldsymbol{x} = (x_1, x_2, \ldots) \in c_0).$$

   Then for each $\boldsymbol{x} \in c_0$, $f_n(\boldsymbol{x}) = x_n \to 0$ as $n \to \infty$. Thus the hypothesis of the proposition is satisfied, with $f(x) = 0$. However, $\|f_n - f\| = \|f_n\| = 1$ for all n, so $f_n \not\to f$.

2. *Let $C[0,1]$ denote the space of all continuous complex-valued functions on $[0,1]$ with the sup norm. Let $P_k \subseteq C[0,1]$ be the subspace of all polynomials of degree at most k. Let $t_0, t_1, \ldots, t_k$ be distinct points in $[0,1]$. Show that if $T : C[0,1] \to P_k$ is a linear mapping such that*

   $$(Tx)(t_i) = x(t_i) \qquad (x \in C[0,1],\ 0 \leqslant i \leqslant k),$$

   then T is continuous.

   Let $R : C[0,1] \to \mathbb{C}^{k+1}$ be the mapping defined by

   $$(Rx)_i = x(t_i) \qquad (0 \leqslant i \leqslant k).$$

   Then R is clearly linear, and is continuous because

   $$\begin{aligned} \|Rx\| &= \max\{|(Rx)_i| : 0 \leqslant i \leqslant k\} \\ &= \max\{|x(t_i)| : 0 \leqslant i \leqslant k\} \\ &\leqslant \max\{|x(t)| : 0 \leqslant t \leqslant 1\} \\ &= \|x\|. \end{aligned}$$

   Define $S : \mathbb{C}^{k+1} \to P_k$ by $S(Rx) = Tx$. Every element of $\boldsymbol{y} \in \mathbb{C}^{k+1}$ is expressible in the form Rx for some $x \in C[0,1]$; this can be done simply by joining the points (t_i, y_i) to form the graph of a (piecewise-linear) function x with $x(t_i) = y_i$ for all i. Moreover, since there is only one polynomial taking any given sequence of values on the points t_i, if $Rx_1 = Rx_2$ then $Tx_1 = Tx_2$. Thus S is well-defined. Since T is linear, S must be linear:

   $$S(\lambda Rx + \mu Ry) = S(R(\lambda x + \mu y)) = T(\lambda x + \mu y) = \lambda Tx + \mu Ty = \lambda S(Rx) + \mu S(Ry).$$

   Since $\mathbb{C}^{k+1}$ is finite-dimensional, the mapping S is continuous. Therefore $T = SR$ is continuous.